Optimal Control Theory

Pontryagin’s Minimum Principle

Pontryagin Minimum Principle

The principle makes the optimal control concept to be effect although the control inputs are constrained and Hamiltonian, H, is not differentiable over $\mathbf{u}$.

Let $\mathbf{u}^*$ is the optimal control and admissible control, $\mathbf{u}$, is bounded.

- If $\mathbf{u}^*$ within the boundary during the entire interval, $[t_0, t_f]$,

$$\delta J(\mathbf{u}^*, \delta\mathbf{u}) = 0$$

- If $\mathbf{u}^*$ lies on the boundary during any position of the time interval,

$$\delta J(\mathbf{u}^*, \delta\mathbf{u}) \geq 0$$

Pontryagin Minimum Principle

The optimal control problem with the constrained control inputs

Minimize $$J(\mathbf{u}(t)) = h(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t)dt$$

Subject to $$\dot{\mathbf{x}}(t) = a(\mathbf{x}(t), \mathbf{u}(t)t)$$ $\mathbf{x}_f, t_f$ free

If δJ_a is obtained as:

$$\delta J_a = \left(\frac{\partial h}{\partial \mathbf{x}} - \boldsymbol{\lambda}^T\right)\delta \mathbf{x}_f + \left(\frac{\partial h}{\partial t} + H\right)\delta t_f$$
$$+ \int_{t_0}^{t_f} \left[\left\{\frac{\partial H}{\partial \mathbf{x}} + \dot{\boldsymbol{\lambda}}^T\right\}\delta \mathbf{x} + \frac{\partial H}{\partial \mathbf{u}}\delta \mathbf{u} + \left(\left[\frac{\partial H}{\partial \boldsymbol{\lambda}}\right]^T - \dot{\mathbf{x}}\right)\delta \boldsymbol{\lambda}\right]dt$$

The necessary conditions and boundary conditions must be satisfied.

$$\delta J(\mathbf{u}^*, \delta \mathbf{u}) \geq 0$$

Pontryagin Minimum Principle

If all other terms are zeros,

$$\delta J_a = \int_{t_0}^{t_f} \left[\frac{\partial H}{\partial \mathbf{u}} \delta \mathbf{u} \right] dt$$

Considering that

$$H(\mathbf{x}^*, \mathbf{u}^* + \delta \mathbf{u}, \boldsymbol{\lambda}^*, t) = H(\mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}^*, t) + \frac{\partial H}{\partial \mathbf{u}} \delta \mathbf{u} + h.o.t.$$

We have

$$\delta J_a = \int_{t_0}^{t_f} [H(\cdots, \mathbf{u}^* + \delta \mathbf{u}, \cdots) - H(\cdots, \mathbf{u}^*, \cdots)] dt \geq 0$$

Therefore, it is necessarily that

$$H(\mathbf{x}^*, \mathbf{u}^* + \delta \mathbf{u}, \boldsymbol{\lambda}^*, t) \geq H(\mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}^*, t) \qquad \forall t$$

Pontryagin Minimum Principle

Necessary Conditions

$$\dot{\mathbf{x}} = \left[\frac{\partial H}{\partial \boldsymbol{\lambda}}\right]^T \longrightarrow \text{State equation}$$

$$\dot{\boldsymbol{\lambda}} = -\left[\frac{\partial H}{\partial \mathbf{x}}\right]^T \longrightarrow \text{Costate equation}$$

$$H(\mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}^*, t) \leq H(\mathbf{x}^*, \mathbf{u}, \boldsymbol{\lambda}^*, t) \longrightarrow \text{Instantaneous optimal condition}$$

for all admissible $\mathbf{u}$

Necessary Conditions at the boundary

$$0 = \left(\frac{\partial h}{\partial \mathbf{x}} - \boldsymbol{\lambda}^T\right)\delta \mathbf{x}_f + \left(\frac{\partial h}{\partial t} + H\right)\delta t_f$$

Minimum Time Control

Consider that there is a state equation of a linear system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

If we are going to find the best control trajectory, $\mathbf{u}^*$, that is constrained by $|\mathbf{u}| \leq \mathbf{u}_{max}$ and minimizes the time when the state is changed from $\mathbf{x}_0$ to $\mathbf{x}_f$. The optimal control problem can be defined as:

$$\text{minimize} \qquad J = \int_{t_0}^{t_f} dt$$

$$\text{subject to} \qquad \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$|\mathbf{u}| \leq \mathbf{u}_{max}$$

Minimum Time Control

Hamiltonian

$$H = 1 + \boldsymbol{\lambda}^T [\mathbf{Ax} + \mathbf{Bu}]$$

Costate equation

$$\dot{\boldsymbol{\lambda}} = -\left[\frac{\partial H}{\partial \mathbf{x}}\right]^T = -\mathbf{A}^T \boldsymbol{\lambda}$$

State equation

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$$

Optimal condition

$$H(\mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}^*, t) \leq H(\mathbf{x}^*, \mathbf{u}^* + \delta\mathbf{u}, \boldsymbol{\lambda}^*, t)$$

$$\boldsymbol{\lambda}^T \mathbf{Bu}^* \leq \boldsymbol{\lambda}^T \mathbf{Bu}$$

Minimum Time Control

Considering that

$$\boldsymbol{\lambda}^T\mathbf{B}\mathbf{u} = [\boldsymbol{\lambda}^T\mathbf{b}_1 \quad \boldsymbol{\lambda}^T\mathbf{b}_2 \quad \dots \quad \boldsymbol{\lambda}^T\mathbf{b}_m]\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \qquad \text{where } \mathbf{B} = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \dots \quad \mathbf{b}_m]$$

The optimal control $\mathbf{u}^*$ is determined as:

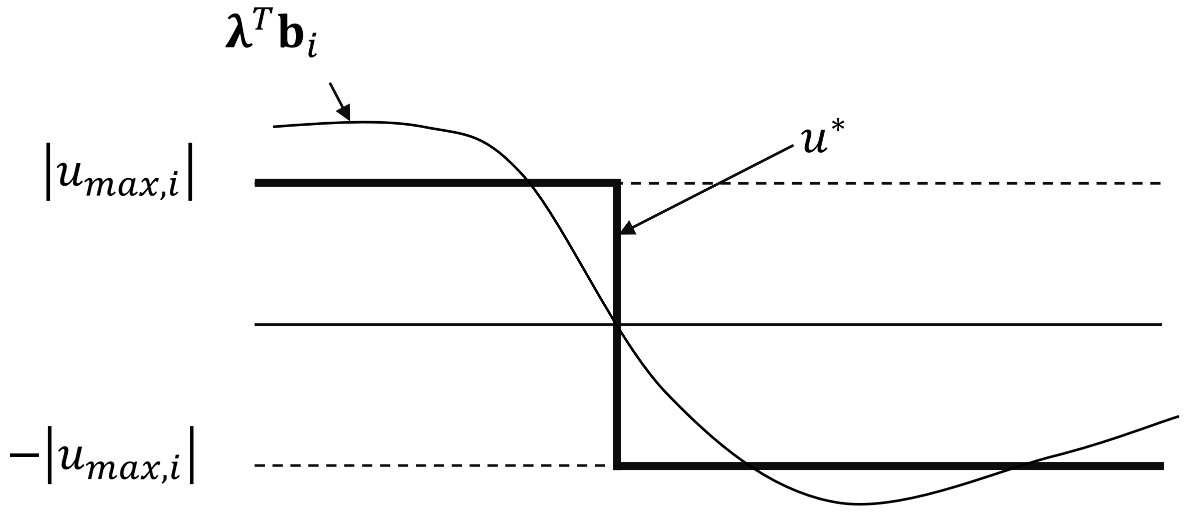
$$\mathbf{u}^* = \begin{bmatrix} u_1^* \\ u_2^* \\ \vdots \\ u_m^* \end{bmatrix} \qquad \text{where } u_i^* = \begin{cases} -sign(\boldsymbol{\lambda}^T\mathbf{b}_i)\left|u_{max,i}\right| & \boldsymbol{\lambda}^T\mathbf{b}_i \neq 0 \\ undetermined & \boldsymbol{\lambda}^T\mathbf{b}_i = 0 \end{cases}$$

The control can be undetermined or singular for a finite time.

Minimum Time Control

The optimal control for the minimum time problem

$$u^* = \begin{cases} |u_{max,i}| & \boldsymbol{\lambda}^T \mathbf{b}_i < 0 \\ -|u_{max,i}| & \boldsymbol{\lambda}^T \mathbf{b}_i > 0 \\ undertermined & \boldsymbol{\lambda}^T \mathbf{b}_i = 0 \end{cases}$$

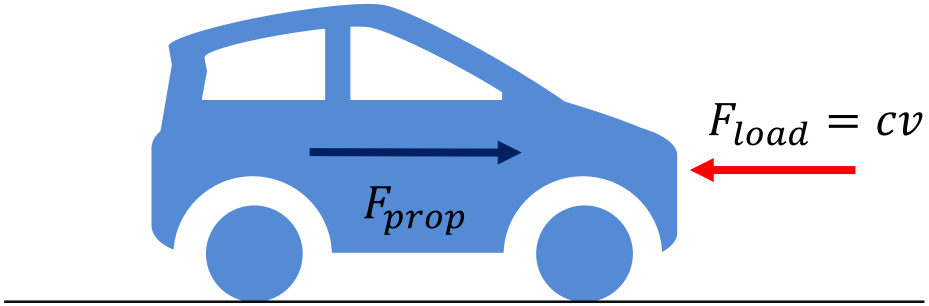


Bang-bang control

Minimum Time Control

Example

Obtain an optimal force trajectory and the minimum time for accelerating from 0 to 100km/h where $F_{prop} < F_{max}$.



From a constitutive equation, $F_{load} = cv$

$$m\dot{v} = F_{prop} - cv \longrightarrow \dot{v} = \frac{1}{m}F_{prop} - \frac{c}{m}v$$

State equation

$$\dot{x} = ax + bu \qquad a = -\frac{c}{m} \qquad b = \frac{1}{m}$$

Minimum Time Control

Hamiltonian

$$H = 1 + \lambda(ax + bu)$$

Costate equation

$$\dot{\lambda} = -\left(\frac{\partial H}{\partial x}\right) = -a\lambda \longrightarrow \lambda = Ce^{-at}$$

State equation

$$\dot{x} = ax + bu \qquad a = -\frac{c}{m} \qquad b = \frac{1}{m}$$

Optimal condition

$$1 + \lambda(ax^* + bu^*) \leq 1 + \lambda(ax^* + b\tilde{u})$$

$$\lambda bu^* \leq \lambda b\tilde{u}$$

$$u^* = \begin{cases} -sign(\lambda b)|u_{max}| & \lambda b \neq 0 \\ undetermined & \lambda b = 0 \end{cases}$$

Minimum Time Control

The final state is fixed.

$$\frac{\partial h}{\partial t} + H = 0 \qquad \text{at } t = t_f \longrightarrow \qquad H(t_f) = 0$$

$$H_f = 1 + \lambda_f (a x_f + b u_f) = 1 + \lambda_f \dot{x}_f$$

by looking the boundary conditions,

$$\dot{x}_f > 0 \longrightarrow \lambda_f < 0 \; for \; \forall t$$

$$\lambda b < 0 \qquad \because b = \frac{1}{m}$$

finally, the optimal control must be

$$u^* = |u_{max}|$$

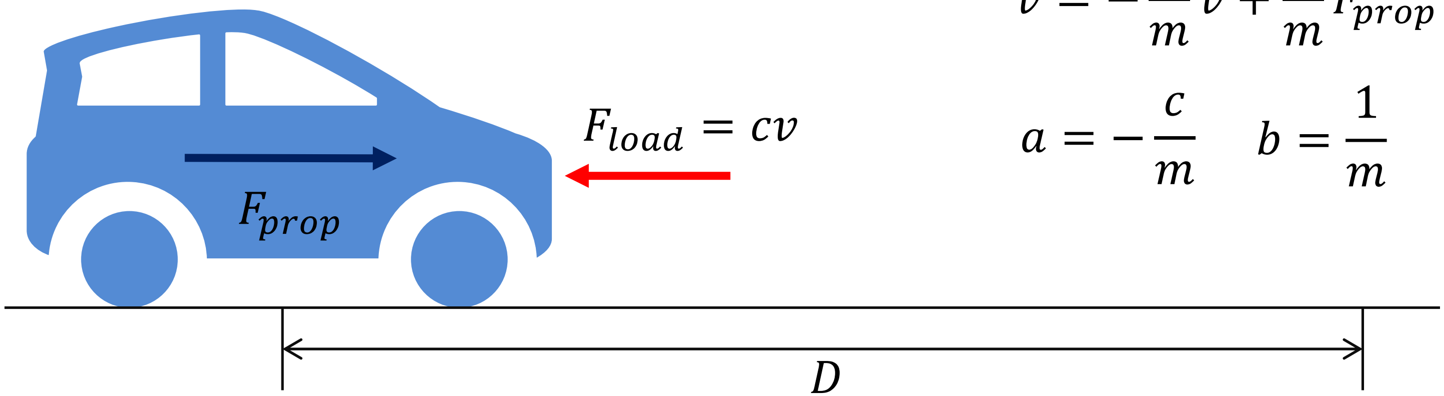
Minimum Time Control

Example

Obtain an optimal force trajectory and the minimum time, so that the car is arrive at the final distance, D, with zero speed, or $0km/h$, where $F_{min} < F_{prop} < F_{max}$

$$\dot{v} = -\frac{c}{m}v + \frac{1}{m}F_{prop}$$

$$a = -\frac{c}{m} \qquad b = \frac{1}{m}$$



State equation

$$\dot{x} = ax + bu$$

$$\dot{z} = x$$

$$z(t) = \int_0^t x(\tau)d\tau$$

$$z(0) = 0$$

$$z(t_f) = D$$

Minimum Time Control

Hamiltonian $H = 1 + \lambda_1(ax + bu) + \lambda_2 x$

Costate equation

$$\dot{\lambda}_1 = -\left(\frac{\partial H}{\partial x}\right) = -a\lambda_1 + \lambda_2$$

$$\dot{\lambda}_2 = -\left(\frac{\partial H}{\partial z}\right) = 0 \longrightarrow \lambda_2 = C \longrightarrow \lambda_1 = C_1 e^{-at} + C_2$$

State equation

$$\dot{x} = ax + bu$$

$$\dot{z} = x$$

Optimal condition

$$1 + \lambda_1(ax^* + bu^*) + \lambda_2 x^* \leq 1 + \lambda(ax^* + b\tilde{u}) + \lambda_2 x^*$$

$$\lambda_1 bu^* \leq \lambda_1 b\tilde{u}$$

$$u^* = \begin{cases} -sign(\lambda_1 b)|u_{max}| & \lambda_1 b \neq 0 \\ undetermined & \lambda_1 b = 0 \end{cases}$$

Minimum Time Control

The final state is fixed.

$$\frac{\partial h}{\partial t} + H = 0 \qquad \text{at } t = t_f \longrightarrow H(t_f) = 0$$

$$H_f = 1 + \lambda_1(t_f)(ax_f + bu_f) + \lambda_2(t_f)x_f = 1 + \lambda_1(t_f)\dot{x}_f = 0 \qquad \because x_f = 0$$

by looking the boundary conditions,

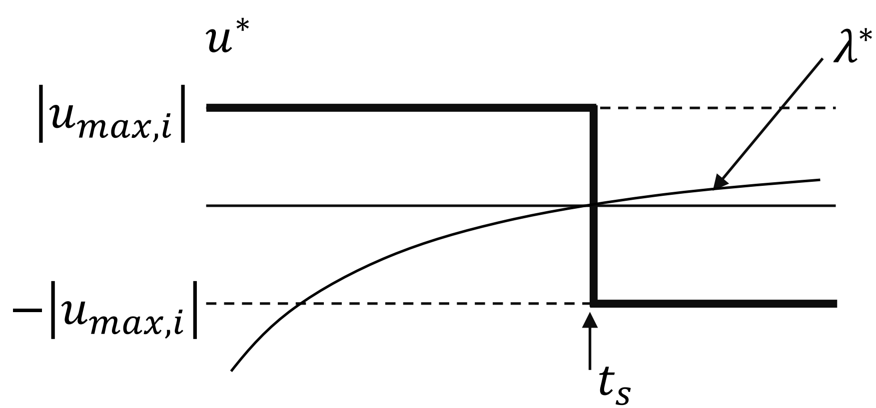
$$\dot{x}_f < 0 \longrightarrow \lambda_1(t_f) > 0$$

$$\lambda_1 = C_1 e^{-at} + C_2 \longrightarrow C_1 \cdot C_2 < 0$$

finally, the optimal can be

$$\lambda_1(t_s) = 0$$

$$\int_0^{t_s} \dot{x}^*(\tau)d\tau = -\int_{t_s}^{t_f} \dot{x}^*(\tau)d\tau$$



Minimum Effort Problem

The objective of the problem is to minimize the effort to control the system where the state satisfies the requirement.

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}, t) + \mathbf{B}(t)\mathbf{u}(t)$$

Find the best control trajectory, $\mathbf{u}^*$, that is constrained by $|\mathbf{u}| \leq \mathbf{u}_{max}$ and minimizes the effort when the state is changed from $\mathbf{x}_0$ to $\mathbf{x}_f$. The optimal control problem can be defined as:

$$\text{minimize} \qquad J = \int_{t_0}^{t_f} \sum_{i=1}^{m} |u_i| \, dt$$

$$\text{subject to} \qquad \dot{\mathbf{x}} = f(\mathbf{x}, t) + \mathbf{B}\mathbf{u}$$

$$|\mathbf{u}| \leq \mathbf{u}_{max}$$

Minimum Effort Problem

Hamiltonian

$$H = \sum_{i=1}^{m} |u_i| + \boldsymbol{\lambda}^T [f(\mathbf{x}, t) + \mathbf{B}(t)\mathbf{u}(t)]$$

Costate equation

$$\dot{\boldsymbol{\lambda}} = -\left[\frac{\partial H}{\partial \mathbf{x}}\right]^T = -\frac{\partial f}{\partial \mathbf{x}}^T \boldsymbol{\lambda}$$

State equation

$$\dot{\mathbf{x}} = f(\mathbf{x}, t) + \mathbf{B}(t)\mathbf{u}(t)$$

Optimal condition

$$H(\mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda}^*, t) \leq H(\mathbf{x}^*, \mathbf{u}^* + \delta\mathbf{u}, \boldsymbol{\lambda}^*, t)$$

$$\sum_{i=1}^{m} |u_i^*| + \boldsymbol{\lambda}^T \mathbf{B}\mathbf{u}^* \leq \sum_{i=1}^{m} |u_i| + \boldsymbol{\lambda}^T \mathbf{B}\mathbf{u}$$

Minimum Effort Problem

The Hamiltonian can be expressed as:

$$H = \sum_{i=1}^{m} |u_i| + \boldsymbol{\lambda}^T \mathbf{b}_i u_i \qquad\qquad i = 1,2,\dots,m$$

At ith control u_i,

If $u_i \geq 0$ $\quad |u_i| + \boldsymbol{\lambda}^T \mathbf{b}_i u_i = (\boldsymbol{\lambda}^T \mathbf{b}_i + 1) u_i$

If $u_i < 0$ $\quad |u_i| + \boldsymbol{\lambda}^T \mathbf{b}_i u_i = (\boldsymbol{\lambda}^T \mathbf{b}_i - 1) u_i$

$$|\mathbf{u}| \leq \mathbf{u}_{max}$$

$$-u_{i,max} \leq u_i \leq u_{i,max}$$

$\boldsymbol{\lambda}^T \mathbf{b}_i > 1 \longrightarrow Ku_i$ where $K > 0 \longrightarrow u_i = -u_{i,max}$

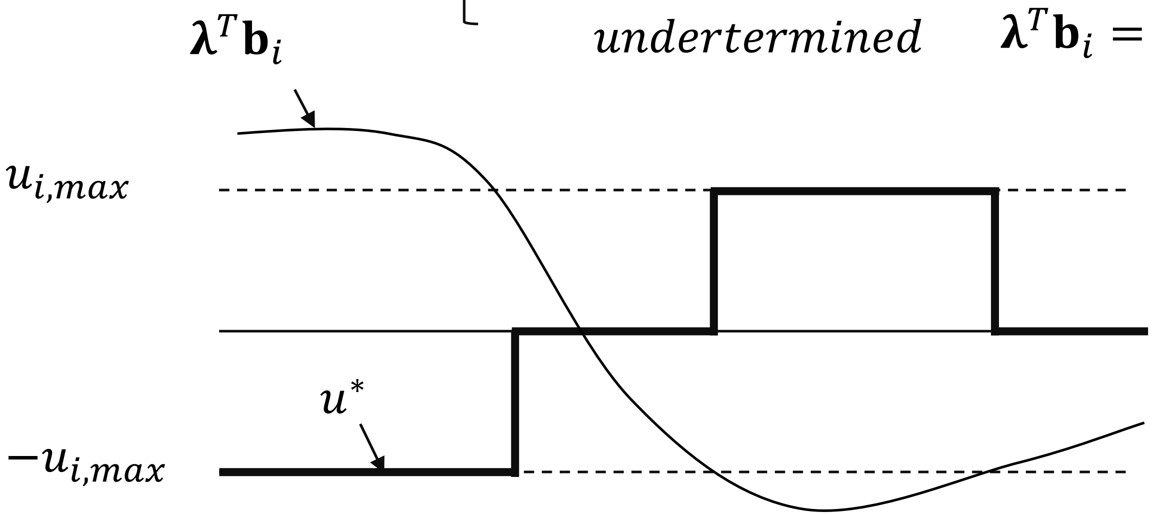
$\boldsymbol{\lambda}^T \mathbf{b}_i < -1 \longrightarrow -Ku_i$ where $K > 0 \longrightarrow u_i = u_{i,max}$

$$-1 < \boldsymbol{\lambda}^T \mathbf{b}_i < 1 \quad \begin{cases} u_i \geq 0, & Ku_i \longrightarrow u_i = 0 \\ u_i < 0, & -Ku_i \longrightarrow u_i = 0 \end{cases}$$

Minimum Effort Problem

The optimal control for the minimum effort problem

$$
u^* = \begin{cases} u_{i,max} & \boldsymbol{\lambda}^T\mathbf{b}_i < -1 \\ 0 & -1 < \boldsymbol{\lambda}^T\mathbf{b}_i < 1 \\ -u_{i,max} & \boldsymbol{\lambda}^T\mathbf{b}_i > 1 \\ undertermined & \boldsymbol{\lambda}^T\mathbf{b}_i = \pm 1 \text{ for a finite time} \end{cases}
$$



Bang-off-bang control