Heat Transfer
DM23815

Chapter 3. One-dimensional steady-state conduction

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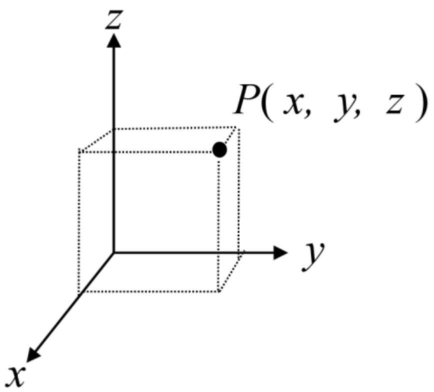
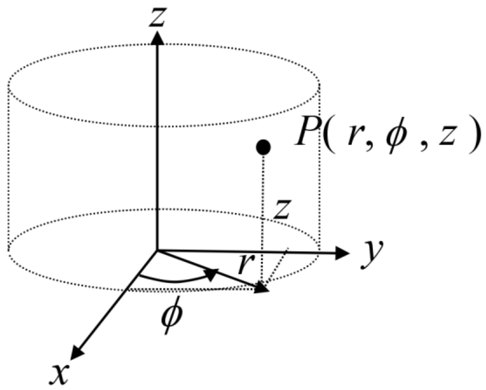
3. Introduction

▪ Coordinate system

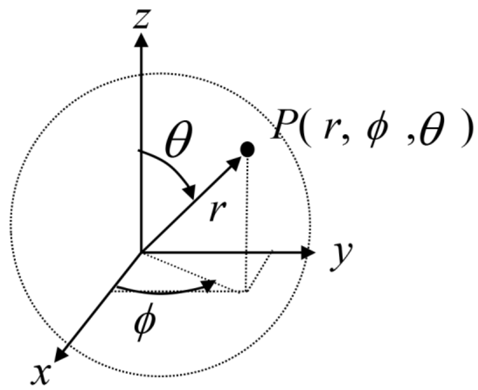
Cartesian coordinates



Cylindrical coordinates



Spherical coordinates



- One dimensional system

Temperature gradients exist along a single coordinate system, and heat transfer occurs **exclusively** in common (planar, cylindrical, and spherical) geometries.

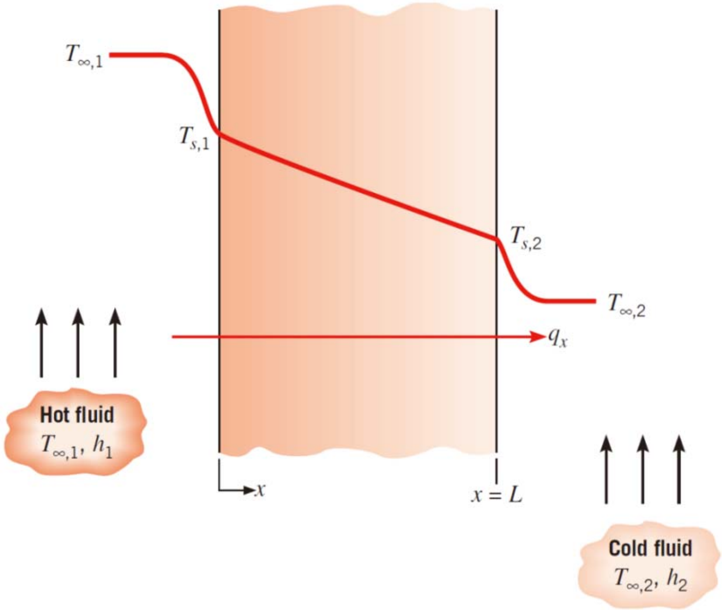
$$\frac{\partial T}{\partial x} >> \frac{\partial T}{\partial y}, \frac{\partial T}{\partial x} >> \frac{\partial T}{\partial z}$$

Under steady-state, one-dimensional conditions with no heat generation

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\cancel{\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)}+\cancel{\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)}+\cancel{\dot{q}}=\rho C\cancel{\frac{\partial T}{\partial t}} \longrightarrow \frac{d}{dx}\left(k\frac{dT}{dx}\right)=0$$

3.1 Plain Wall

▪ Temperature Distribution



constant thermal conductivity

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right)=0$$

$$\int d\left(\frac{dT}{dx}\right)=\int 0dx \quad \Rightarrow \quad \frac{dT}{dx}=C_1$$

$$\int dT=\int C_1 dx$$

To obtain two unknown constants, two boundary conditions T(0) = $T_{s,1}$ and T(L) = $T_{s,2}$ are necessary.

at $x = 0$ $\quad T_{s,1} = C_2$

$$T(x) = (T_{s,2} - T_{s,1})\frac{x}{L} + T_{s,1}$$

For one-dimensional, steady-state with no heat generation and constant conductivity, the **temperature** varies **linearly** with *x*.

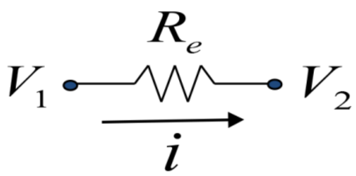
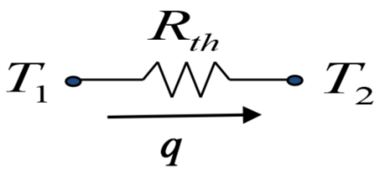
$$q_x = -kA\frac{dT}{dx} = \frac{kA}{L}(T_{s,1} - T_{s,2})$$

$$q''_x = \frac{q_x}{A} = \frac{k}{L}(T_{s,1} - T_{s,2})$$

3.1 Plain Wall

▪ Analogy Analysis

Resistance: Ratio of a driving potential to the corresponding transfer rate

Electricity		**Heat transfer**	
	Voltage: $(V_2\text{-}V_1)$		Thermal potential: $(T_2\text{-}T_1)$
	Current: i		Heat flow: q
$R_e = \dfrac{V_1 - V_2}{i}$	Resistance: R_{e}	$R_{th} = \dfrac{T_1 - T_2}{q}$	Thermal resistance: R_{th}

Conduction Case

$$q_x = \frac{kA}{L}\left(T_{s,1} - T_{s,2}\right)$$

$$R_{t,cond} \equiv \frac{T_{s,1} - T_{s,2}}{q_x} = \frac{L}{kA}$$

Convection Case

$$q = hA\left(T_s - T_\infty\right)$$

$$R_{t,conv} \equiv \frac{T_s - T_\infty}{q} = \frac{1}{hA}$$

Radiation Case

$$q = \varepsilon\sigma\, A\left(T_s^{\,4} - T_{\mathrm{sur}}^{\,4}\right)$$

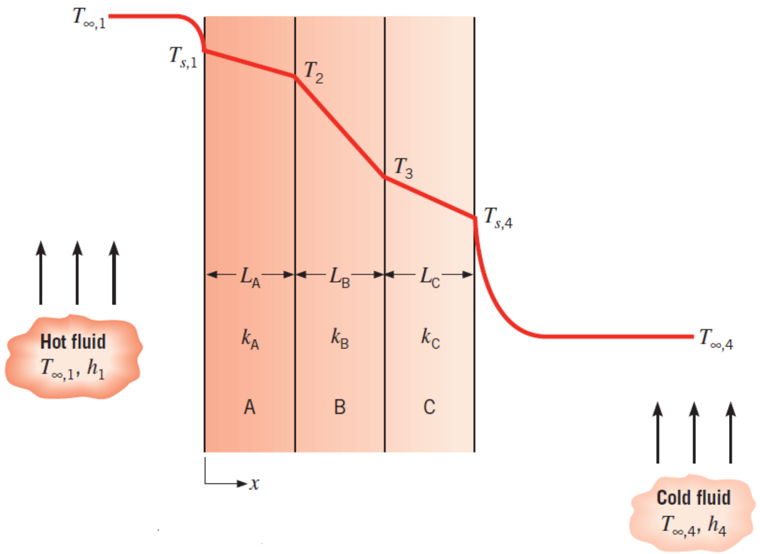
$$q = \varepsilon\sigma\, A\left(T_s^{\,2} + T_{\mathrm{sur}}^{\,2}\right)\left(T_s + T_{\mathrm{sur}}\right)\left(T_s - T_{\mathrm{sur}}\right)$$

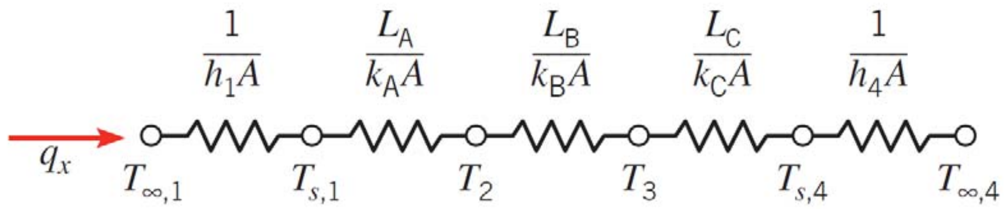
$$q = h_{rad} A\left(T_s - T_{\mathrm{sur}}\right)$$

$$R_{t,\mathrm{rad}} \equiv \frac{T_s - T_{\mathrm{sur}}}{q} = \frac{1}{h_{\mathrm{rad}} A}$$

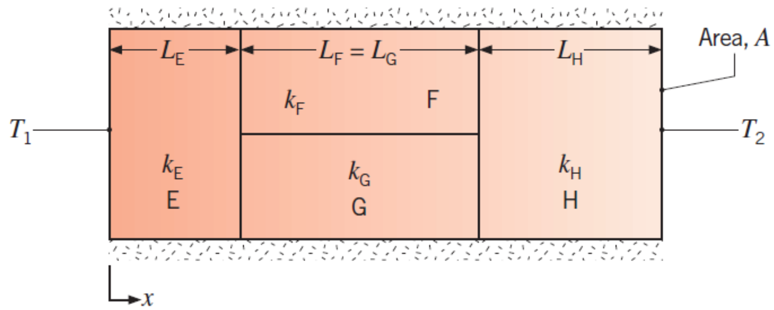
3.1 Plain Wall

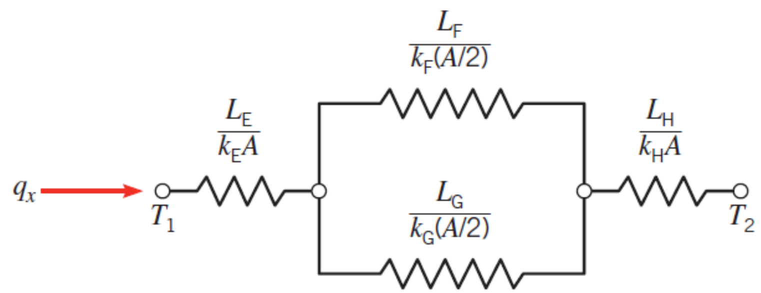
▪ Steady heat conduction in multilayer plane wall





$$q_x = \frac{T_{\infty,1} - T_{\infty,4}}{[(1/h_1 A) + (L_A/k_A A) + (L_B/k_B A) + (L_C/k_C A) + (1/h_4 A)]}$$



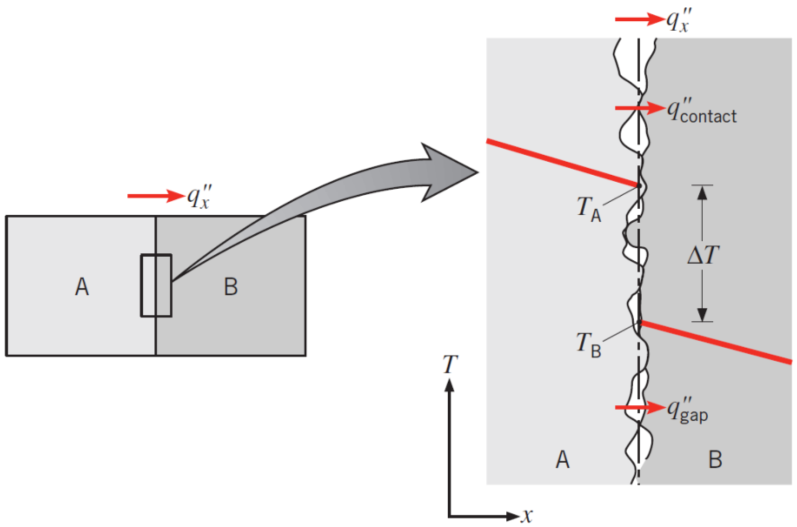


Overall heat transfer coefficient(열관류계수), U $\qquad q_x = UA\Delta T$

$$U = \frac{1}{R_{tot}A} = \frac{1}{[(1/h_1) + (L_A/k_A) + (L_B/k_B) + (L_C/k_C) + (1/h_4)]}$$

3.1 Plain Wall

▪ Thermal Contact Resistance



- The existence of a finite contact resistance is due principally to **surface roughness effects**.
- Contact spots are interspersed with gaps that are, in most instances, **air filled**. Heat transfer is therefore due to **conduction** across the **actual contact area** and to **conduction** and/or **radiation** across **the gaps**.
- The contact resistance may also be reduced by selecting an **interfacial fluid** of large thermal conductivity.

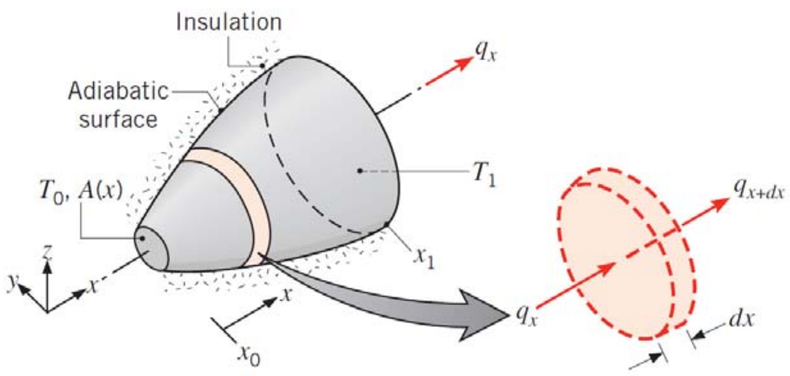
Thermal contact resistance for (a) metallic interfaces under vacuum conditions and (b) aluminum interface

Thermal Resistance, $R''_{t,c} \times 10^4$ ($m^2 \cdot K/W$)				
(a) Vacuum Interface			**(b) Interfacial Fluid**	
Contact pressure	100 kN/m^2	10,000 kN/m^2	Air	2.75
Stainless steel	6–25	0.7–4.0	Helium	1.05
Copper	1–10	0.1–0.5	Hydrogen	0.720
Magnesium	1.5–3.5	0.2–0.4	Silicone oil	0.525
Aluminum	1.5–5.0	0.2–0.4	Glycerine	0.265

Thermal resistance of representative solid/solid interfaces

Interface	$R''_{t,c} \times 10^4$ ($m^2 \cdot K/W$)
Silicon chip/lapped aluminum in air (27–500 kN/m^2)	0.3–0.6
Aluminum/aluminum with indium foil filler (~100 kN/m^2)	~0.07
Stainless/stainless with indium foil filler (~3500 kN/m^2)	~0.04
Aluminum/aluminum with metallic (Pb) coating	0.01–0.1
Aluminum/aluminum with Dow Corning 340 grease (~100 kN/m^2)	~0.07
Stainless/stainless with Dow Corning 340 grease (~3500 kN/m^2)	~0.04
Silicon chip/aluminum with 0.02-mm epoxy	0.2–0.9
Brass/brass with 15-μm tin solder	0.025–0.14

3.2 Alternative Conduction Analysis



- Under, **steady-state** conditions with **no heat generation** and no **heat loss** from the sides, heat transfer rate q_x must be a constant independent of x.
- Even if the area varies with position A(x) and the thermal conductivity varies with temperature k(T), $q_x = q_{x+dx}$.

$$q_x \int_{x_0}^{x} \frac{dx}{A(x)} = -\int_{T_0}^{T} k(T)\, dT$$

T_0 is known, $T = T_1$ at some $x = x_1$ is known.

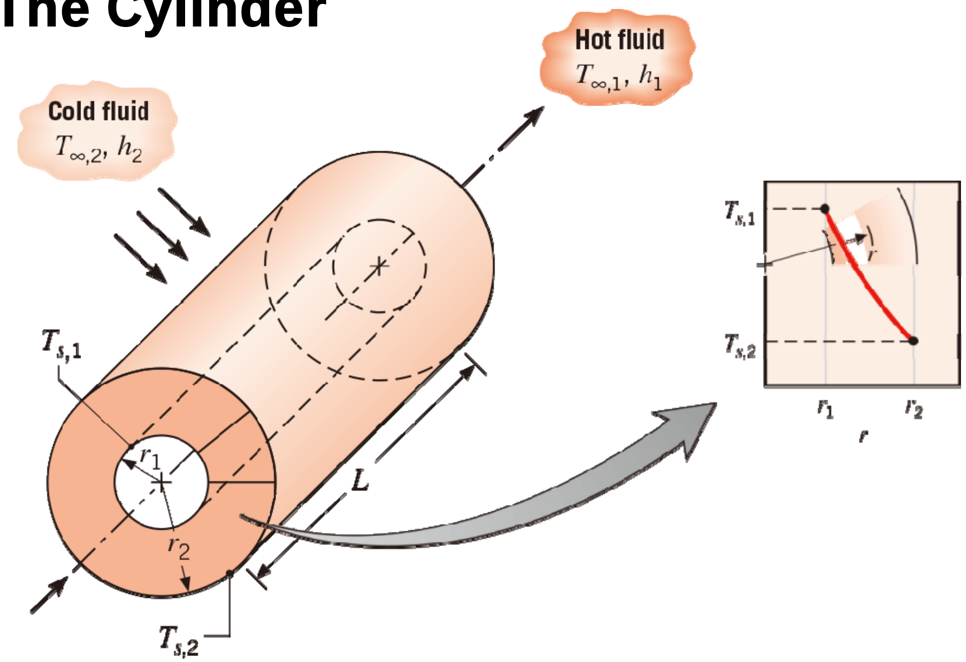
q_x may be computed by integrating between x_0 and x_1.

A is uniform and k is independent of temperature, above equation reduces to

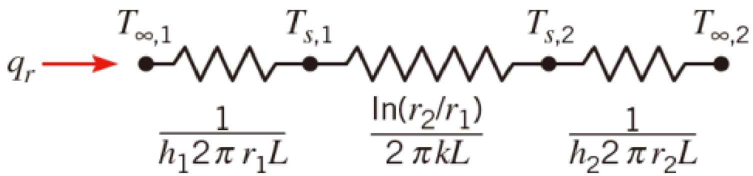
$$\frac{q_x\, \Delta x}{A} = -k\, \Delta T$$

3.3 Radial Systems

- **The Cylinder**



$$q_r = -kA\frac{dT}{dr} = -k(2\pi rL)\frac{dT}{dr}$$



For steady-state conditions with no heat generation

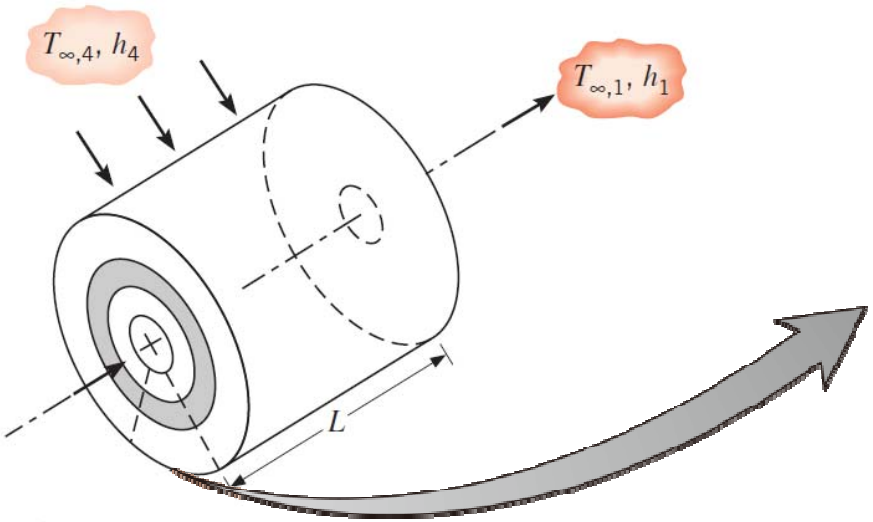
$$\frac{1}{r}\frac{d}{dr}\left(kr\frac{dT}{dr}\right) = 0$$

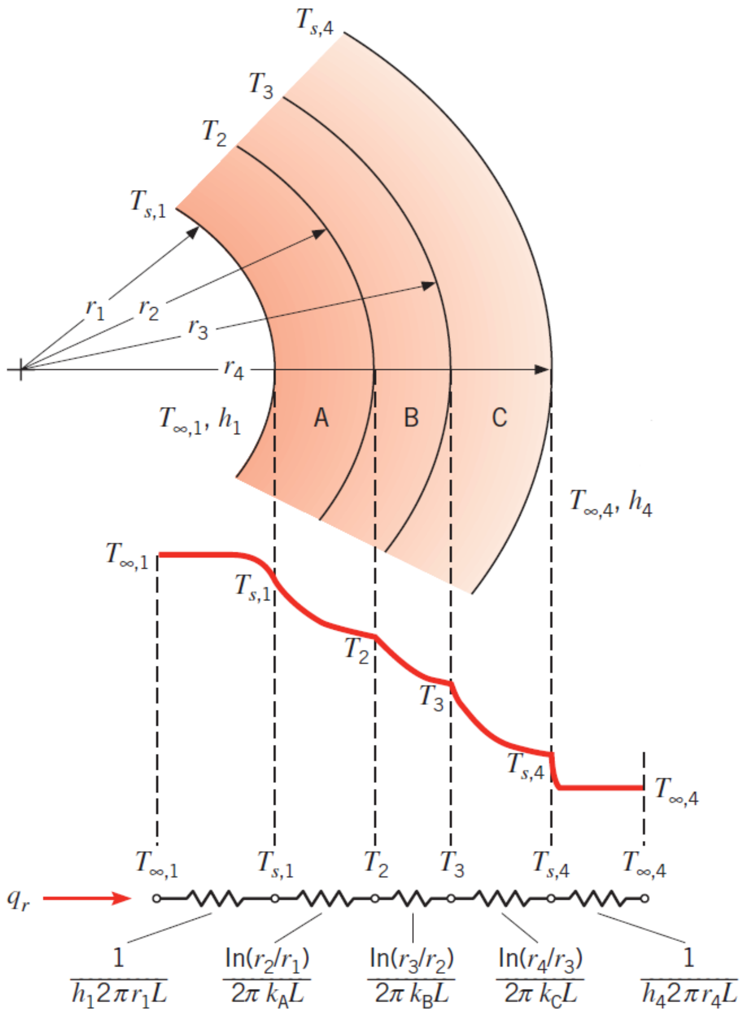
$$T(r_1) = T_{s,1} \quad \text{and} \quad T(r_2) = T_{s,2} \qquad T_{s,1} = C_1 \ln r_1 + C_2 \qquad T_{s,2} = C_1 \ln r_2 + C_2$$

$$T_{s,1} - T_{s,2} = C_1 \ln r_1 - C_1 \ln r_2 \qquad C_1 = \frac{T_{s,1} - T_{s,2}}{\ln r_1 - \ln r_2} = \frac{T_{s,1} - T_{s,2}}{\ln\left(\frac{r_1}{r_2}\right)} \qquad C_2 = T_{s,2} - \frac{(T_{s,1} - T_{s,2})}{\ln\left(\frac{r_1}{r_2}\right)} \ln r_2$$

3.3 Radial Systems

- **Multilayered cylinder**





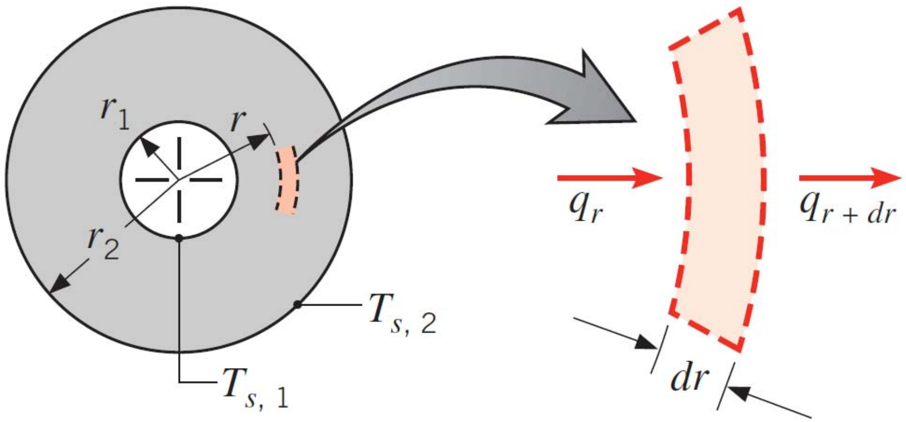
$$q_r = \frac{T_{\infty,1} - T_{\infty,4}}{\frac{1}{2\pi r_1 L h_1} + \frac{\ln(r_2/r_1)}{2\pi k_A L} + \frac{\ln(r_3/r_2)}{2\pi k_B L} + \frac{\ln(r_4/r_3)}{2\pi k_C L} + \frac{1}{2\pi r_4 L h_4}}$$

$$q_r = \frac{T_{\infty,1} - T_{\infty,4}}{R_{tot}} = UA(T_{\infty,1} - T_{\infty,4})$$

$$U = \frac{1}{\frac{1}{h_1} + \frac{r_1}{k_A}\ln\frac{r_2}{r_1} + \frac{r_1}{k_B}\ln\frac{r_3}{r_2} + \frac{r_1}{k_C}\ln\frac{r_4}{r_3} + \frac{r_1}{r_4}\frac{1}{h_4}}$$

3.3 Radial Systems

- **The Sphere**



$$q_r = -kA\frac{dT}{dr} = -k(4\pi r^2)\frac{dT}{dr}$$

$$q_x \int_{x_0}^{x} \frac{dx}{A(x)} = -\int_{T_0}^{T} k(T)\, dT$$

$$q_r = \frac{4\pi k(T_{s,1} - T_{s,2})}{(1/r_1) - (1/r_2)}$$

$$R_{t,\text{cond}} = \frac{1}{4\pi k}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

- **Summary**

	Plane Wall	Cylindrical Wall[a]	Spherical Wall[a]
Heat equation	$\frac{d^2T}{dx^2} = 0$	$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$	$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = 0$
Temperature distribution	$T_{s,1} - \Delta T\frac{x}{L}$	$T_{s,2} + \Delta T\frac{\ln(r/r_2)}{\ln(r_1/r_2)}$	$T_{s,1} - \Delta T\left[\frac{1-(r_1/r)}{1-(r_1/r_2)}\right]$
Heat flux (q'')	$k\frac{\Delta T}{L}$	$\frac{k\Delta T}{r\ln(r_2/r_1)}$	$\frac{k\Delta T}{r^2[(1/r_1)-(1/r_2)]}$
Heat rate (q)	$kA\frac{\Delta T}{L}$	$\frac{2\pi Lk\Delta T}{\ln(r_2/r_1)}$	$\frac{4\pi k\Delta T}{(1/r_1)-(1/r_2)}$
Thermal resistance ($R_{t,\text{cond}}$)	$\frac{L}{kA}$	$\frac{\ln(r_2/r_1)}{2\pi Lk}$	$\frac{(1/r_1)-(1/r_2)}{4\pi k}$

3.5 Conduction with Thermal Energy Generation

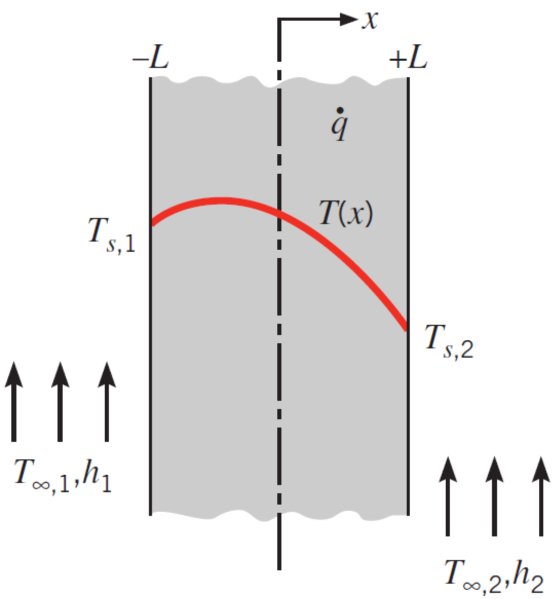
$$E_g = I^2 \cdot R_e$$

$$\dot{q} \equiv \frac{E_g}{V} = \frac{I^2 R_e}{V}$$

(W/m^3)

- The **temperature distribution** of processes may be **occurring** within the medium due to thermal energy generation.
- A common thermal energy generation process involves the conversion from **electrical** to **thermal energy** in a **current-carrying medium** (Ohmic, or resistance, or Joule heating).
- Energy generation may also occur as a result of the **deceleration** and **absorption** of **neutrons** in the fuel element of a nuclear reactor or exothermic **chemical reactions**.

▪ The Plane Wall



For constant thermal conductivity k and uniform energy generation per unit volume ($\dot{q}$ is constant)

$$\frac{d^2T}{dx^2} + \frac{\dot{q}}{k} = 0$$

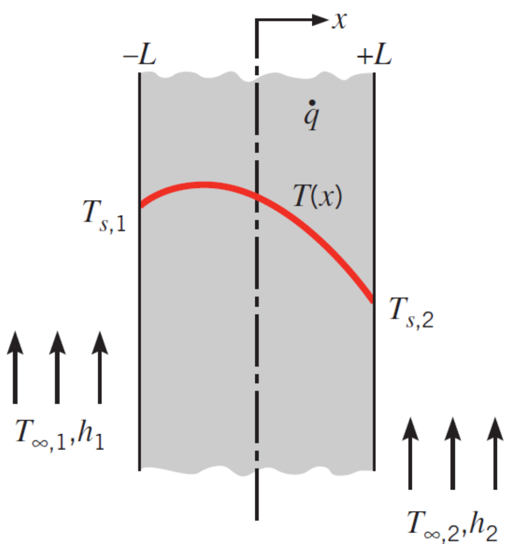
The general solution is

where C_1 and C_2 are the constants of integration.

C_1 and C_2 are determined through **boundary conditions**.

3.5 Conduction with Thermal Energy Generation

- **Asymmetrical Boundary Conditions**



The general solution is $T = -\dfrac{\dot{q}}{2k}x^2 + C_1 x + C_2$

$$T(-L) = T_{s,1} \qquad T(L) = T_{s,2}$$

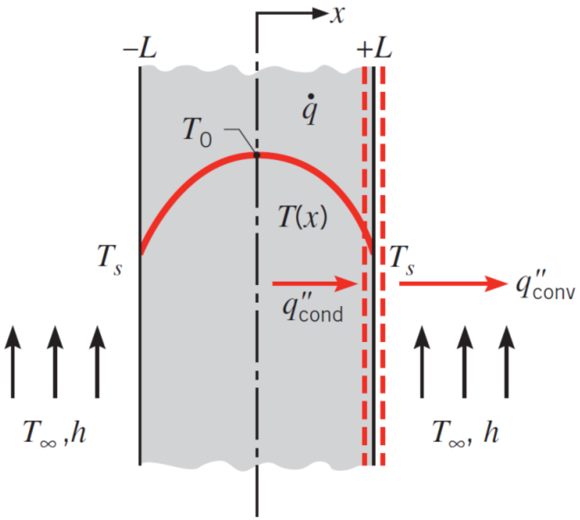
$$x = -L \qquad T = T_{s,1} = -\frac{\dot{q}L^2}{2k} - C_1 L + C_2$$

$$x = L \qquad T = T_{s,2} = -\frac{\dot{q}L^2}{2k} + C_1 L + C_2$$

$$T(x) = \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + \frac{T_{s,2} - T_{s,1}}{2}\frac{x}{L} + \frac{T_{s,1} + T_{s,2}}{2}$$

3.5 Conduction with Thermal Energy Generation

▪ **Symmetrical Boundary Conditions**

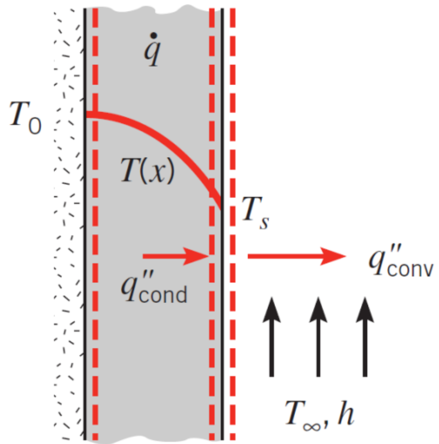


The general solution is $$T = -\frac{\dot{q}}{2k}x^2 + C_1 x + C_2$$

$$T_{s,1} = T_{s,2} \equiv T_s$$

$x = -L$

$x = L$



$$T(x) = \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + T_s$$

The maximum temperature exists at the midplane

$$T(0) \equiv T_0 = \frac{\dot{q}L^2}{2k} + T_s$$