The Helmholtz Equation

Substituting $U(\mathbf{r},t) = U(\mathbf{r})\exp(j2\pi\nu t)$ from (2.2-5) into the wave equation

$$\boxed{\nabla^2 U + k^2 U = 0\,,}$$

Helmholtz Equation
(Wave equation for monochromatic wave)

$$k = \frac{2\pi\nu}{c} = \frac{\omega}{c}$$ **wavenumber**

Optical intensity (using complex amplitude)

$$\begin{aligned} 2u^2(\mathbf{r},t) &= 2a^2(\mathbf{r})\cos^2[2\pi\nu t + \varphi(\mathbf{r})] \\ &= |U(\mathbf{r})|^2\{1 + \cos(2[2\pi\nu t + \varphi(\mathbf{r})])\} \end{aligned}$$

$$\boxed{I(\mathbf{r}) = |U(\mathbf{r})|^2\,.}$$

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Elementary waves

Plane waves : solution of Helmholtz equation in Cartesian coordinate

$$\boxed{\nabla^2 U + k^2 U = 0\,,}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$U(\mathbf{r}) = A\exp(-j\mathbf{k}\cdot\mathbf{r}) = A\exp\left[-j(k_x x + k_y y + k_z z)\right]$$

Equal-phase surface (wavefront) is given as

$$k_x x + k_y y + k_z z = \text{constant}$$

a plane having normal vector of $k_x\hat{x} + k_y\hat{y} + k_z\hat{z}$

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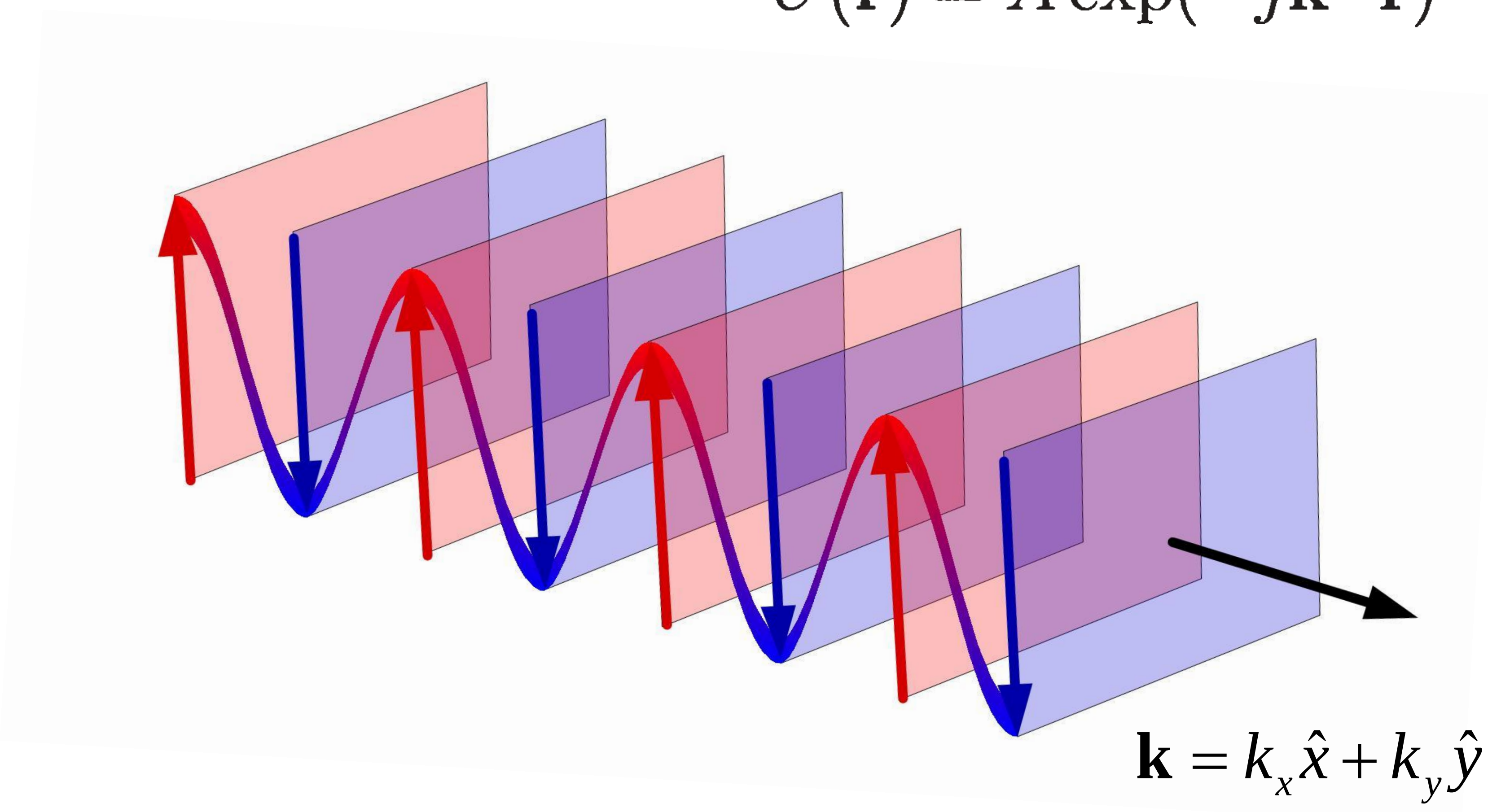
Plane waves

$$U(\mathbf{r}) = A\exp(-j\mathbf{k}\cdot\mathbf{r})$$



$$\mathbf{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z}$$

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Plane waves

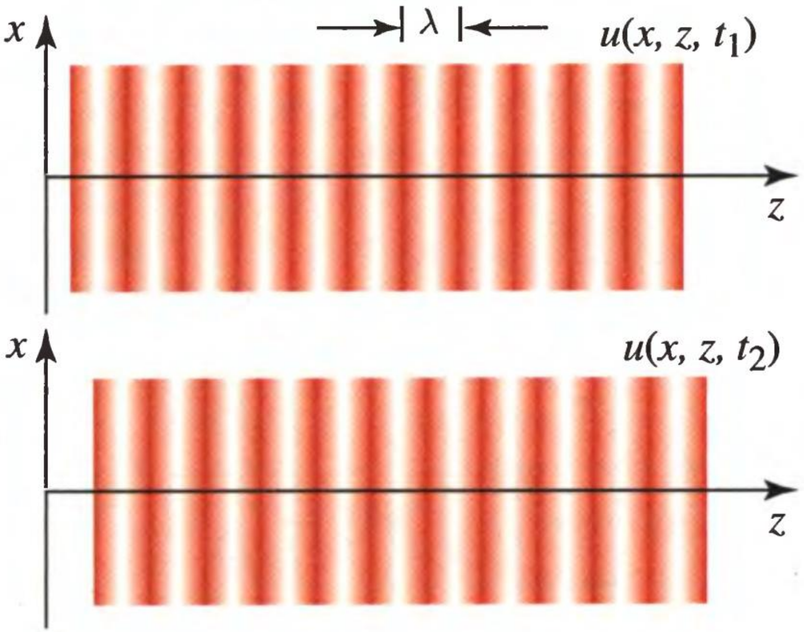
Plane wave propagating through *z* direction

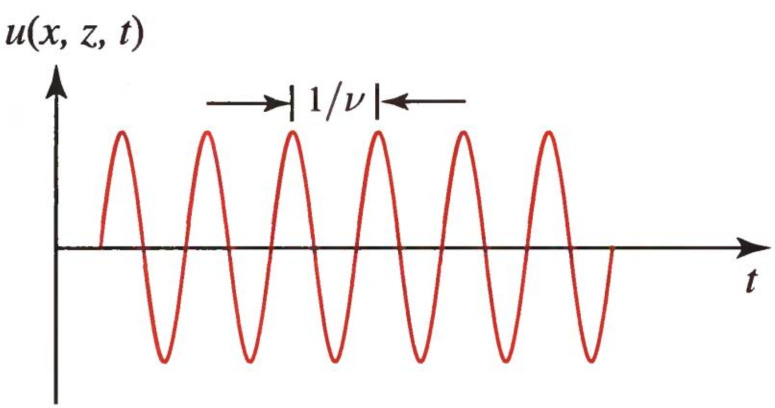
$$\mathbf{k} = k\hat{z} \qquad U(\mathbf{r}) = A\exp(-jkz)$$

$$\lambda = \frac{c}{\nu}$$

$$u(\mathbf{r}, t) = |A| \cos\left[2\pi\nu(t - z/c) + \arg\{A\}\right]$$

phase velocity





Spherical waves

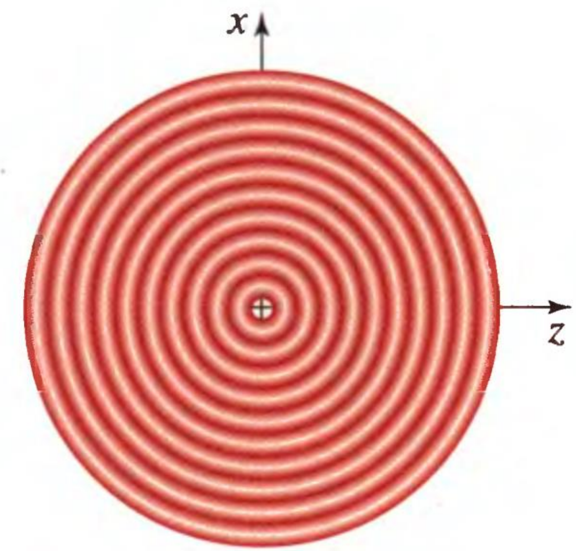
Solution of Helmholtz equation in Spherical coordinate
(with radial symmetry)

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) + k^2 U = 0 \xrightarrow[\text{Hint}]{} U(\mathbf{r}) = \frac{A_0}{r}\exp(-jkr)$$

$$V(r) = rU(r)$$

H.W.



Spherical Wave

$$\boxed{U(\mathbf{r}) = \frac{A_0}{r}\exp(-jkr)}$$

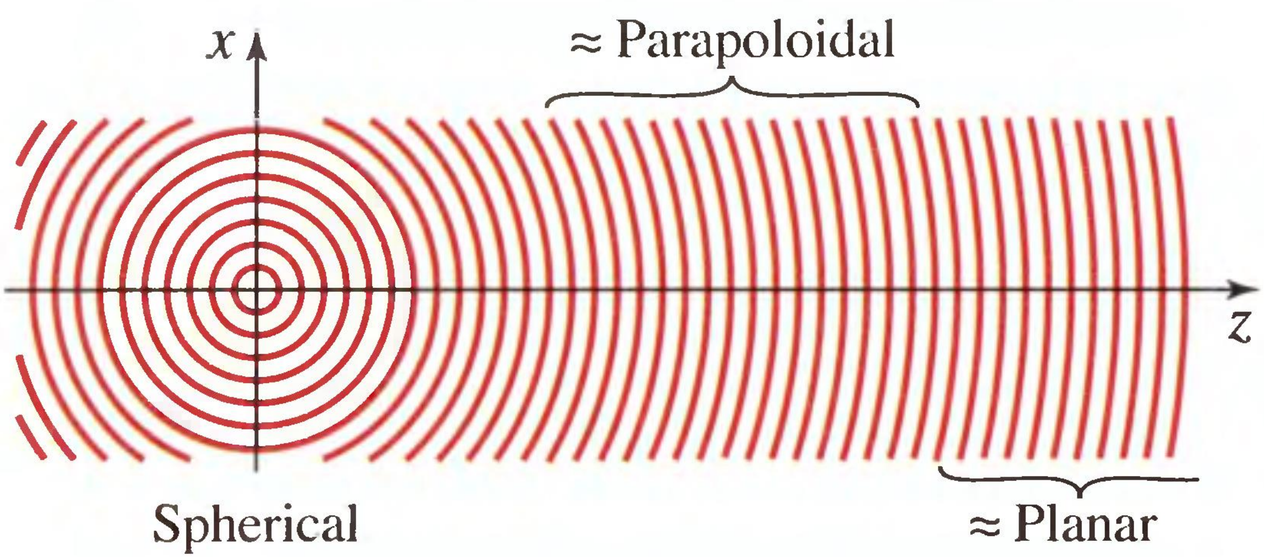
Paraboloidal waves

Fresnel Approximation of Spherical wave

- Related to paraxial approximation of ray-optics

$$\sqrt{x^2+y^2} \ll z \qquad r \approx z + (x^2+y^2)/2z$$

$$U(\mathbf{r}) = \frac{A_0}{r}\exp(-jkr) \longrightarrow U(\mathbf{r}) \approx \frac{A_0}{z}\exp(-jkz)\exp\left[-jk\frac{x^2+y^2}{2z}\right].$$



Fresnel Approximation of a Spherical Wave

Condition for Fresnel approx.

Third term of Taylor expansion must be smaller than pi

$$z\left(1+\frac{\theta^2}{2}-\frac{\theta^4}{8}+\cdots\right) \longrightarrow kz\theta^4/8 \ll \pi,$$

$$\frac{N_F\theta_m^2}{4} \ll 1$$

Condition for applying Fresnel approximation

– <u>Light emitted from point source can be considered as paraboloidal wave when this condition is satisfied.</u>

$$N_F = \frac{a^2}{\lambda z}$$

Fresnel Number

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Paraxial waves

A paraxial wave is a wave that can be generally written as,

$$U(\mathbf{r}) = A(\mathbf{r}) \exp(-jkz)$$

Therefore, paraboloidal wave is also one of paraxial waves.

$$U(\mathbf{r}) \approx \frac{A_0}{z} \exp(-jkz) \exp\left[-jk\frac{x^2+y^2}{2z}\right]. \quad A(\mathbf{r})$$

- Optical wavefront mainly changed along *z* axis.
- Complex amplitude A(r) slowly varying along *z* axis

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Paraxial Helmholtz equation

Helmholtz equation

$$\nabla^2 U + k^2 U = 0\,,$$

Substitute $\longleftarrow$ $U(\mathbf{r}) = A(\mathbf{r}) \exp(-jkz)$

Paraxial Helmholtz equation

$$\nabla_T^2 A - j\,2k\frac{\partial A}{\partial z} = 0 \qquad \nabla_T^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$$

Paraxial Helmholtz equation is a slowly varying envelope approximation of the Helmholtz equation.

- Plane wave, spherical wave : Solution of exact Helmholtz Eq.
- Paraboloidal wave, Gaussian beam (Ch.3) :
Solution of paraxial Helmholtz Eq.

Fundamental of Photonics
Wave-Optics (2)

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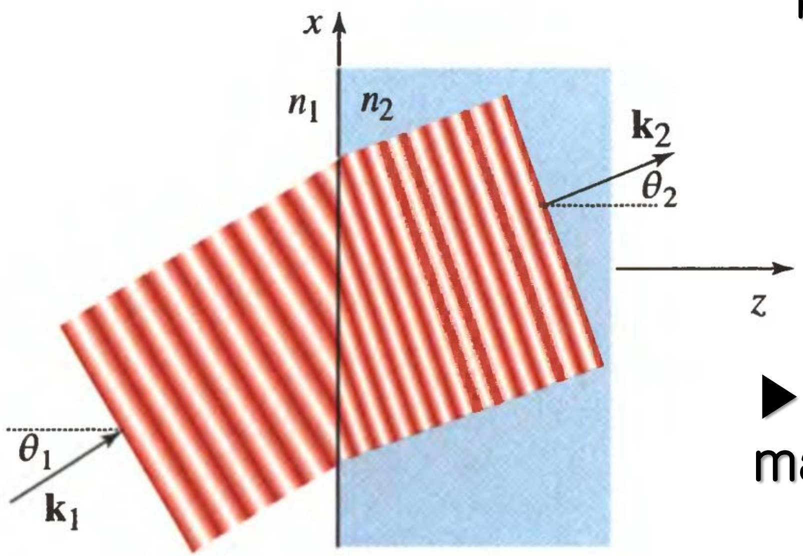
Optical components (Wave Opt.)

▶ Reflection & Refraction

: Phase must be same for incident, reflected, and refracted light at the interface.



phase matching condition

$$\mathbf{k}_1 \cdot \mathbf{r} = \mathbf{k}_2 \cdot \mathbf{r} = \mathbf{k}_3 \cdot \mathbf{r} \qquad \text{for all } \mathbf{r} = (x, y, 0)$$

$$k_1 = k_3 = n_1 k_o \text{ and } k_2 = n_2 k_0$$

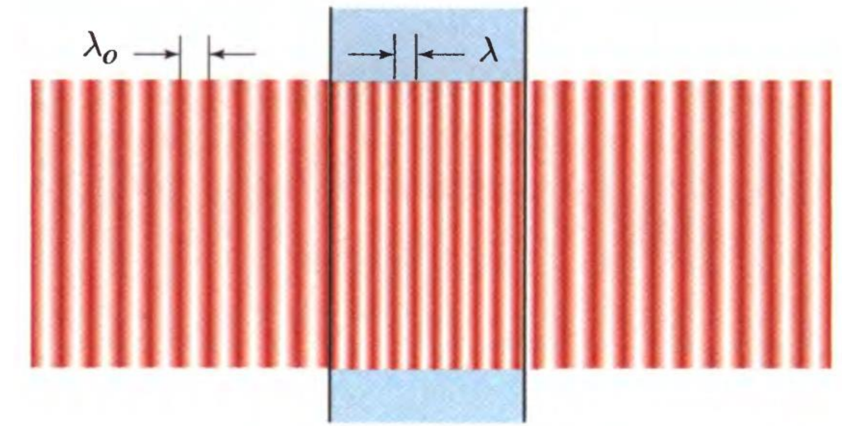
▶ Snell's law can be derived by phase matching condition

$$k_{1x} = k_{2x}$$

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

Thin optical elements

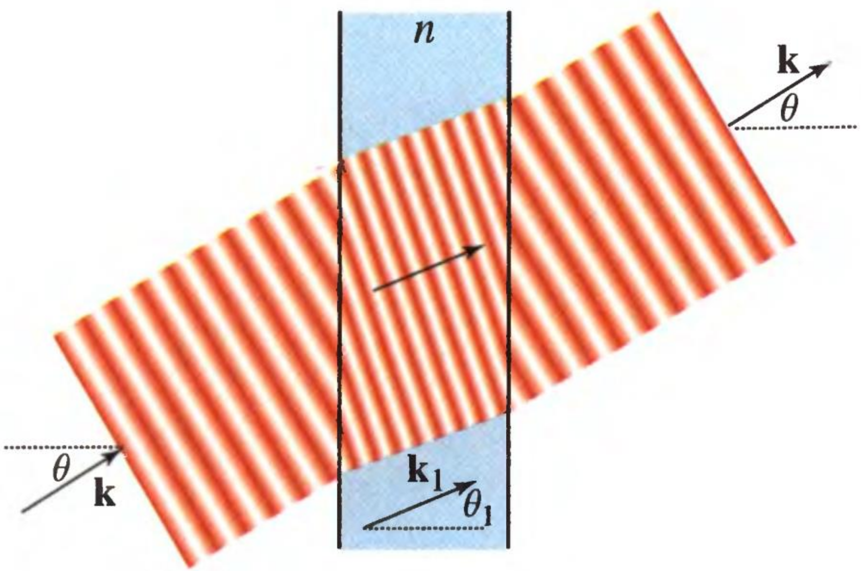
▶ Phase delay caused by thin optical elements



$$U(x, y, d)/U(x, y, 0)$$

$$t(x, y) = \exp(-jnk_o d)\,.$$

Transmittance of flat dielectric plate (normal incidence)



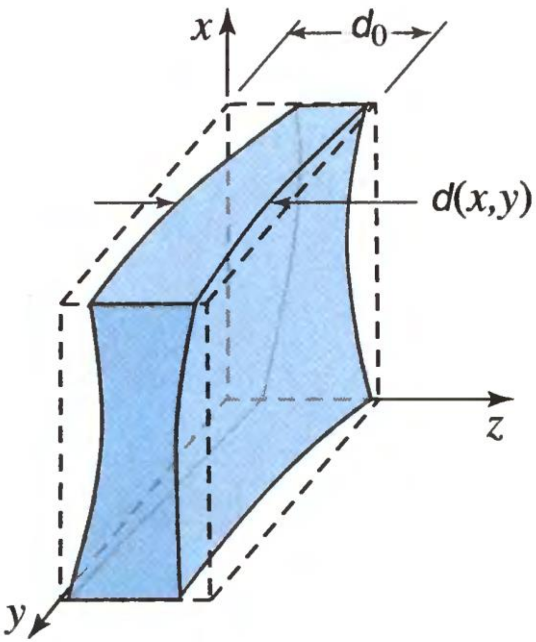
$$\exp(-j\mathbf{k}_1 \cdot \mathbf{r}) \;=\; \exp[-jnk_o(z\cos\theta_1 + x\sin\theta_1)]$$

$$t(x, y) = \exp\left(-jnk_o d\,\cos\theta_1\right)$$

Transmittance of flat dielectric plate (oblique incidence)

Thin optical elements

▶ Thin plate with variable thickness (refractive index = n)



Optical path length: $nd(x,y)+n_{air}(d_0-d(x,y))$

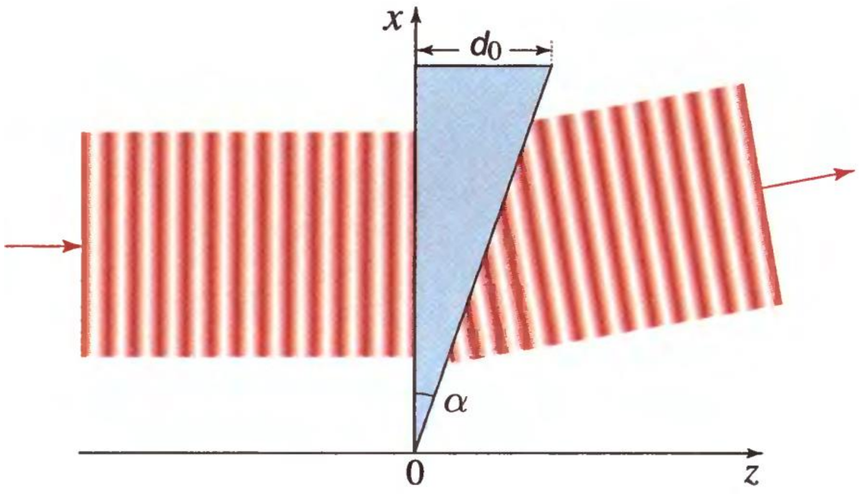
Phase delay:

$$t(x,y) \approx \exp[-jnk_o d(x,y)]\exp[-jk_o(d_0-d(x,y))]$$

$$t(x,y) \approx h_0 \exp[-j(n-1)k_o d(x,y)]$$

Transmittance of thin plate with variable thickness
(slowly varying thickness)

Thin prism



When alpha is small,

$$t(x,y) = h_0 \exp[-j(n-1)k_o \alpha x]$$

Plane wave incidence

$$U(x,y,z) = A\exp(-jk_0 z) \qquad z<0$$

Outgoing wave (paraxial approximation)

$$U(x,y,z) = Ah_0 \exp(-j(n-1)\alpha k_0 x)\exp(-jk_z z) \qquad z>d$$

Outgoing wave have wavevector of $\mathbf{k} = \left((n-1)\alpha k_0, 0, k_z\right)$

$$k_z = \sqrt{k_0^{\ 2} - k_x^{\ 2}} \approx k_0 \qquad \sin\theta_d \approx \theta_d \approx (n-1)\alpha$$

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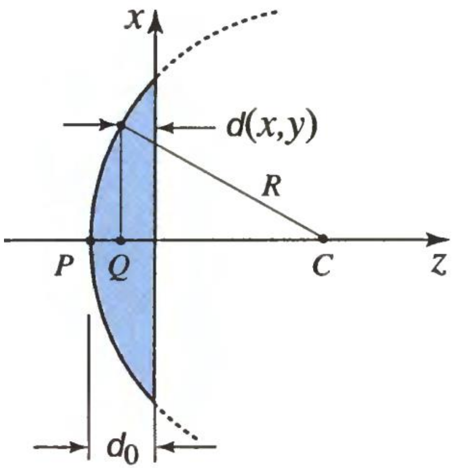
Thin lens



$$d(x,y) = d_0 - \left[R - \sqrt{R^2 - (x^2+y^2)}\right]$$

$$\sqrt{R^2 - (x^2+y^2)} \approx R\left(1 - \frac{x^2+y^2}{2R^2}\right)$$

$$d(x,y) \approx d_0 - \frac{x^2+y^2}{2R}$$

Transmittance of thin planoconvex lens

$$t(x,y) \approx h_0 \exp\left[jk_o \frac{x^2+y^2}{2f}\right] \qquad f = \frac{R}{n-1}$$

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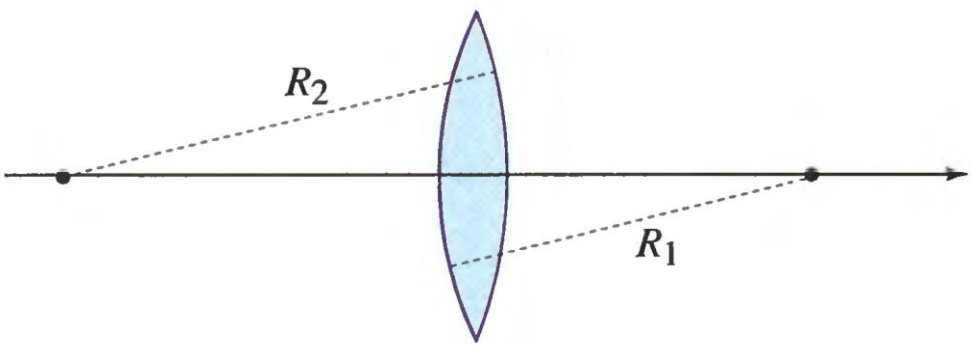
Thin lens



A (double) convex lens

$$t_{R_1}(x,y) = h_1 \exp(jk_0 \frac{x^2+y^2}{2f_1})$$

$$t_{R_2}(x,y) = h_2 \exp(jk_0 \frac{x^2+y^2}{2f_2})$$

$$t(x,y) = h_1 h_2 \exp(jk_0 \frac{(x^2+y^2)}{2}\left(\frac{1}{f_1}+\frac{1}{f_2}\right)) = h_0 \exp(jk_0 \frac{(x^2+y^2)}{2f})$$

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} = (n-1)\left(\frac{1}{R_1} - \frac{1}{(-R_2)}\right)$$

Same fomula with ray-optic Assumption!

Thin lens

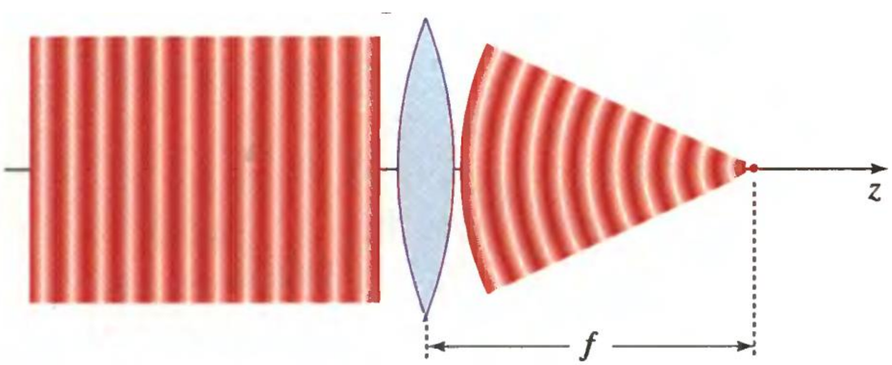
Plane wave incidence

$$U(x,y,z) = A\exp(-jk_0 z) \qquad z<0$$

Outgoing wave (paraxial approximation)

Wavefront Φ

$$U(x,y,z) \approx Ah_0 \exp\left(-jk_0 \frac{-x^2}{2f}\right)\exp(-jk_0 z) = C\exp\left(-j\boxed{k_0\left(\frac{-x^2}{2f}+z\right)}\right) \qquad z>d$$



Wavevector at position x_0

$$\nabla\Phi = \mathbf{k} = \left(-k_0 x_0 / f, 0, k_0\right)$$

It focused to focal point!