Helmholtz equation

Substituting $U(\mathbf{r}, t) = U(\mathbf{r})\exp(j2\pi\nu t)$ into the wave equation

$$\boxed{(\nabla^2 + k^2)U(\mathbf{r}) = 0,} \qquad k = \frac{2\pi\nu}{c} = \frac{\omega}{c}$$ (wavenumber)

: Helmholtz equation

➡ ***"The wave equation for monochromatic waves"***

The optical intensity

$$I(\mathbf{r}) = |U(\mathbf{r})|^2$$

$$I(\mathbf{r}, t) = 2\langle u^2(\mathbf{r}, t)\rangle$$

$$2u^2(\mathbf{r}, t) = 2a^2(\mathbf{r})\cos^2[2\pi\nu t + \varphi(r)]$$

$$= |U(\mathbf{r})|^2\{1 + \cos(2[2\pi\nu t + \varphi(\mathbf{r})])\}$$

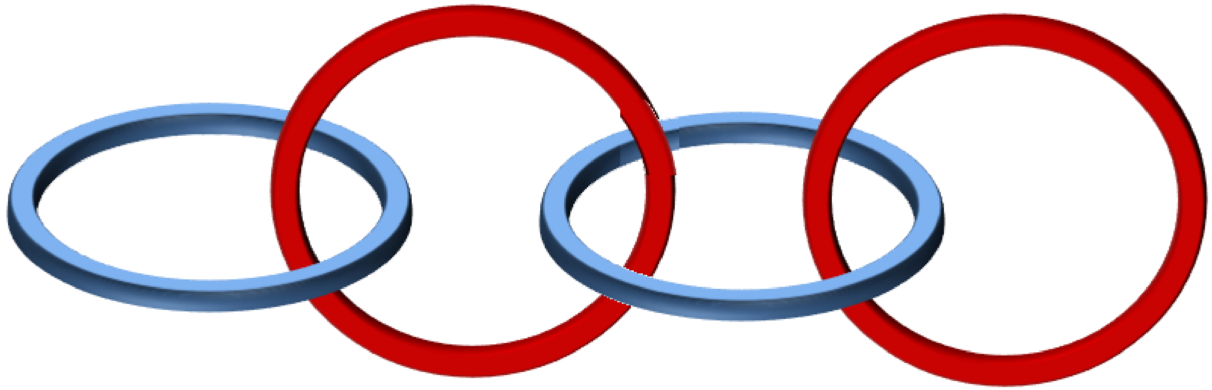
The intensity of a monochromatic wave does not vary with time.

Gauss's law $\nabla \cdot \mathbf{D} = \rho_V$

Faraday's law $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$

No magnetic charges $\nabla \cdot \mathbf{B} = 0$
(Gauss's law for magnetism)

Ampère's law $\nabla \times \mathbf{H} = \mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}$



$$\frac{d^2y(x)}{dx^2} = -k^2y(x)$$

IF $\quad \frac{\partial E(x,t)}{\partial x} = -\frac{\partial B(x,t)}{\partial \mathrm{t}} \quad$? $\quad \partial x$

$$\frac{\partial B(x,t)}{\partial x} = +\frac{\partial E(x,t)}{\partial \mathrm{t}} \qquad \partial \mathrm{t}$$

$$\frac{\partial^2y(x,t)}{\partial x^2} = \frac{1}{v}^{2} \frac{d^2y(x,t)}{dt^2} \qquad , w = 2\pi f, k = \frac{2\pi}{\lambda}$$

$$\frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} = \frac{\partial^2\psi}{\partial x^2}$$

Trial solution

$$\psi(x,t) = \mathrm{X}(x)\mathrm{T}(t)$$

$$\mathrm{X}(x)\frac{1}{v^2}\frac{\partial^2\mathrm{T}(t)}{\partial t^2} = \mathrm{T}(t)\frac{\partial^2\mathrm{X}(x)}{\partial x^2} \ldots (1)$$

$$\frac{1}{\mathrm{T}(t)}\frac{1}{v^2}\frac{\partial^2\mathrm{T}(t)}{\partial t^2} = \frac{1}{\mathrm{X(x)}}\frac{\partial^2\mathrm{X}(x)}{\partial x^2} \ldots (2)$$

$$\frac{1}{\mathrm{T}}\frac{1}{v^2}\frac{\partial^2\mathrm{T}}{\partial t^2} = -k^2 \qquad \ldots (1)$$

$$\frac{1}{\mathrm{X}}\frac{\partial^2\mathrm{X}}{\partial x^2} = -k^2 \qquad \ldots (2)$$

$$k^2v^2 = \omega^2$$

$$\frac{\partial^2\mathrm{T}}{\partial t^2} = -\omega^2\mathrm{T} \ldots (1)$$

$$\frac{\partial^2\mathrm{X}}{\partial x^2} = -k^2X \ldots (2)$$

cos(A+B) = cosAcosB - sinAsinB
sin(A+B) = sinAcosB + cosAsinB

$$\mathrm{T}(t) = \mathrm{A}\cos(\omega \mathrm{t}) + \mathrm{B}\sin(\omega \mathrm{t})$$
$$= \left(\sqrt{A^2 + \mathrm{B}^2}\right)\cos(\omega\mathrm{t} + \phi_0),\ \tan\phi_0 = -\frac{B}{A}$$
$$= A'\cos(\omega\mathrm{t} + \phi_0)$$

$$\mathrm{X}(x) = \mathrm{C}\cos(kx) + \mathrm{D}\sin(kx)$$
$$= \left(\sqrt{C^2 + \mathrm{D}^2}\right)\cos(kx + \phi_1),\ \tan\phi_0 = -\frac{D}{C}$$
$$= B'\cos(kx + \phi_1)$$

복소해?

$$\mathrm{T}(t) = A'\exp(j(\omega\mathrm{t} + \phi_0))$$

$$\mathrm{X}(x) = B'\exp(j(kx + \phi_1))$$

$$\mathrm{T}(t)\mathrm{X}(x) = A'\exp(j(kx + \omega\mathrm{t} + \phi_2))$$

$$\frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \nabla^2\psi$$

$$\psi(x, y, t) = X(x)Y(y)T(t)$$

$$T'' + \omega^2 T = 0 \qquad \omega^2 = (k_x^2 + k_y^2)v^2$$

$$X'' + k_x^2 X = 0$$

$$Y'' + k_y^2 Y = 0$$

$$\mathbf{E}(x,y,z;t) = Re[\tilde{\mathbf{E}}(x,y,z)e^{j\omega t}]$$

$$\mathbf{H}(x,y,z;t) = Re[\widetilde{\mathbf{H}}(x,y,z)e^{j\omega t}]$$

$$\left\{\begin{matrix} \mathbf{E} \\ \mathbf{H} \end{matrix}\right\}$$

$$\nabla \cdot \tilde{\mathbf{E}} = \frac{\tilde{\boldsymbol{\rho}}_v}{\varepsilon}$$

$$\nabla \times \tilde{\mathbf{E}} = -\mathrm{j}\omega\mu\widetilde{\boldsymbol{H}}$$

$$\nabla \cdot \widetilde{\mathbf{H}} = 0$$

$$\nabla \times \widetilde{\mathbf{H}} = \tilde{\mathbf{J}} + \mathrm{j}\omega\varepsilon\tilde{\mathbf{E}}$$

Maxwell equations in vacuum

$$\left\{ \begin{aligned} &\nabla \cdot \vec{\mathrm{E}} = \cancel{\frac{\rho_{free}}{\epsilon}} = 0 \\ &\nabla \times \vec{\mathrm{E}} = -\mu \frac{\partial \vec{\mathrm{H}}}{\partial \mathrm{t}} \\ &\nabla \cdot \vec{\mathrm{B}} = 0 \\ &\nabla \times \vec{\mathrm{H}} = \cancel{\mathrm{J}_{\mathrm{free}}} + \varepsilon \frac{\partial \vec{\mathrm{E}}}{\partial \mathrm{t}} \end{aligned} \right.$$

$$\left\{ \begin{aligned} &\nabla \cdot \vec{\mathrm{E}} = 0 \\ &\nabla \times \vec{\mathrm{E}} = -\mu_0 \frac{\partial \vec{\mathrm{H}}}{\partial \mathrm{t}} \\ &\nabla \cdot \vec{\mathrm{B}} = 0 \\ &\nabla \times \vec{\mathrm{H}} = \varepsilon_0 \frac{\partial \vec{\mathrm{E}}}{\partial \mathrm{t}} \end{aligned} \right.$$

$$\left\{ \begin{aligned} &\nabla \times (\nabla \times \vec{\mathrm{E}}) \\ &= -\mu_0 \frac{\partial (\nabla \times \vec{\mathrm{H}})}{\partial \mathrm{t}} \\ &\nabla \times \vec{\mathrm{H}} = \varepsilon_0 \frac{\partial \vec{\mathrm{E}}}{\partial \mathrm{t}} \end{aligned} \right.$$

$$\boxed{\nabla \times (\nabla \times \vec{\mathrm{E}}) = -\mu_0 \varepsilon_0 \frac{\partial \vec{\mathrm{E}}}{\partial^2 \mathrm{t}}}$$

$$\boxed{\nabla^2 \vec{\mathrm{E}} = \mu_0 \varepsilon_0 \frac{\partial \vec{\mathrm{E}}}{\partial^2 \mathrm{t}}}$$

3D wave equation

$$\nabla \times \left(\nabla \times \vec{\mathrm{E}}\right) = \nabla\left(\nabla \cdot \vec{\mathrm{E}}\right) - \nabla^2 \vec{\mathrm{E}}$$

$$\frac{1}{v^2} = \mu_0 \varepsilon_0$$

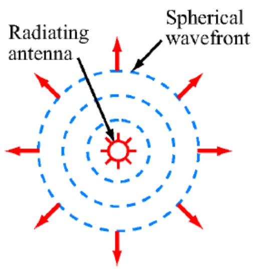
In materials, not trial

$$\begin{cases} \nabla \cdot \vec{\mathrm{E}} = \dfrac{\rho_{free}}{\epsilon} = 0 \\ \nabla \times \vec{\mathrm{E}} = -\mu \dfrac{\partial \vec{\mathrm{H}}}{\partial \mathrm{t}} \\ \nabla \cdot \vec{\mathrm{B}} = 0 \\ \nabla \times \vec{\mathrm{H}} = \mathrm{J}_{\mathrm{free}} + \varepsilon \dfrac{\partial \vec{\mathrm{E}}}{\partial \mathrm{t}} \end{cases}$$

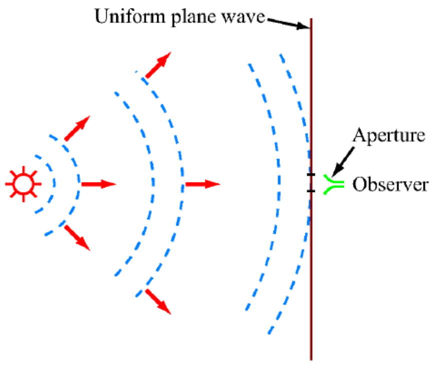
Time-Harmonic Fields

$$\mathbf{E}(x,y,z;t) = Re[\tilde{\mathbf{E}}(x,y,z)e^{j\omega t}]$$

$$\mathbf{H}(x,y,z;t) = Re[\tilde{\mathbf{H}}(x,y,z)e^{j\omega t}]$$



(a) Spherical wave



(b) Plane-wave approximation

$$\left\{ \begin{array}{c} \mathbf{E} \\ \mathbf{H} \end{array} \right\}$$

$$\nabla \cdot \tilde{\mathbf{E}} = \frac{\tilde{\rho}_v}{\varepsilon}$$

$$\nabla \times \tilde{\mathbf{E}} = -j\omega\mu\tilde{\boldsymbol{H}}$$

$$\nabla \cdot \tilde{\mathbf{H}} = 0$$

$$\nabla \times \tilde{\mathbf{H}} = \tilde{\mathbf{J}} + j\omega\varepsilon\tilde{\mathbf{E}}$$

Time-Harmonic Fields

$$\nabla \times \tilde{\mathbf{H}} = \tilde{\mathbf{J}} + j\omega\varepsilon\tilde{\mathbf{E}}$$

$$= (\boxed{\qquad} + j\omega\varepsilon)\tilde{\mathbf{E}} = j\omega(\boxed{\qquad\qquad})\tilde{\mathbf{E}}$$

complex permittivity $\varepsilon_c =$

$$\nabla \times \tilde{\mathbf{H}} = j\omega\varepsilon_c\tilde{\mathbf{E}}$$

$$\boxed{\qquad\qquad\qquad\qquad} \quad \cdots \quad (1)$$

H.W

$$\nabla \cdot \nabla \times \tilde{\mathbf{H}} = 0 \quad \rightarrow \nabla \cdot \left(j\omega\varepsilon_c\tilde{\mathbf{E}}\right) = 0 \text{ or } \nabla \cdot \tilde{\mathbf{E}} \rightarrow \widetilde{\rho_v} = 0$$

$$\nabla \cdot \tilde{\mathbf{E}} = \frac{\tilde{\rho}_v}{\varepsilon}$$
$$\nabla \times \tilde{\mathbf{E}} = -j\omega\mu\tilde{\boldsymbol{H}}$$
$$\nabla \cdot \tilde{\mathbf{H}} = 0$$
$$\nabla \times \tilde{\mathbf{H}} = \tilde{\mathbf{J}} + j\omega\varepsilon\tilde{\mathbf{E}}$$

$$\nabla \cdot \tilde{\mathbf{E}} =$$
$$\nabla \times \tilde{\mathbf{E}} =$$
$$\nabla \cdot \tilde{\mathbf{H}} =$$
$$\nabla \times \tilde{\mathbf{H}} =$$

Time-Harmonic Fields

$$\nabla \times \tilde{\mathbf{H}} = \tilde{\mathbf{J}} + j\omega\varepsilon\tilde{\mathbf{E}}$$
$$= (\sigma + j\omega\varepsilon)\tilde{\mathbf{E}} = j\omega\left(\varepsilon - j\frac{\sigma}{\omega}\right)\tilde{\mathbf{E}}$$

complex permittivity $\varepsilon_c = (\varepsilon - j\dfrac{\sigma}{\omega})$

$$\boxed{\nabla \times \tilde{\mathbf{H}} = j\omega\varepsilon_c\tilde{\mathbf{E}}} \quad \cdots \quad (1)$$

H.W

$$\nabla \cdot \nabla \times \tilde{\mathbf{H}} = 0 \quad \rightarrow \nabla \cdot \left(j\omega\varepsilon_c\tilde{\mathbf{E}}\right) = 0 \text{ or } \nabla \cdot \tilde{\mathbf{E}} \rightarrow \widetilde{\rho_v} = 0$$

$$\begin{aligned} \nabla \cdot \tilde{\mathbf{E}} &= \frac{\tilde{\boldsymbol{\rho}}_v}{\varepsilon} \\ \nabla \times \tilde{\mathbf{E}} &= -j\omega\mu\tilde{\boldsymbol{H}} \\ \nabla \cdot \tilde{\mathbf{H}} &= 0 \\ \nabla \times \tilde{\mathbf{H}} &= \tilde{\mathbf{J}} + j\omega\varepsilon\tilde{\mathbf{E}} \end{aligned} \quad \Rightarrow \quad \begin{aligned} \nabla \cdot \tilde{\mathbf{E}} &= 0 \\ \nabla \times \tilde{\mathbf{E}} &= -j\omega\mu\tilde{\boldsymbol{H}} \\ \nabla \cdot \tilde{\mathbf{H}} &= 0 \\ \nabla \times \tilde{\mathbf{H}} &= j\omega\varepsilon_c\tilde{\mathbf{E}} \end{aligned}$$

Time-Harmonic Fields

$$\nabla \cdot \tilde{\mathbf{E}} = 0$$
$$\nabla \times \tilde{\mathbf{E}} = -j\omega\mu\tilde{\boldsymbol{H}}$$
$$\nabla \cdot \tilde{\mathbf{H}} = 0$$
$$\nabla \times \tilde{\mathbf{H}} = j\omega\varepsilon_c\tilde{\mathbf{E}}$$

$$\nabla \times \left(\nabla \times \tilde{\mathbf{E}}\right) =$$
$$\nabla \times \left(\nabla \times \tilde{\mathbf{E}}\right) =$$
$$\nabla \times \left(\nabla \times \tilde{\mathbf{E}}\right) =$$

H.W

$$\nabla^2\tilde{\mathbf{E}} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2}\right)\tilde{\mathbf{E}}$$

∇^2: Laplacian operator

$$\nabla^2\tilde{\mathbf{E}} + \omega^2\mu\varepsilon_c\tilde{\mathbf{E}} = 0$$

$$\gamma_c^2 = -\omega^2\mu\varepsilon_c$$

Propagation constant

Time-Harmonic Fields

$$\nabla \cdot \tilde{\mathbf{E}} = 0$$
$$\nabla \times \tilde{\mathbf{E}} = -j\omega\mu\tilde{\boldsymbol{H}}$$
$$\nabla \cdot \tilde{\mathbf{H}} = 0$$
$$\nabla \times \tilde{\mathbf{H}} = j\omega\varepsilon_c\tilde{\mathbf{E}}$$

$$\nabla \times (\nabla \times \tilde{\mathbf{E}}) = -j\omega\mu(\nabla \times \tilde{\mathbf{H}})$$
$$\nabla \times \left(\nabla \times \tilde{\mathbf{E}}\right) = -j\omega\mu\left(j\omega\varepsilon_c\tilde{\mathbf{E}}\right) = \omega^2\mu\varepsilon_c\tilde{\mathbf{E}}$$
$$\nabla \times \left(\nabla \times \tilde{\mathbf{E}}\right) = \nabla\left(\nabla \cdot \tilde{\mathbf{E}}\right) - \nabla^2\tilde{\mathbf{E}}$$

H.W

$$\nabla^2\tilde{\mathbf{E}} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2}\right)\tilde{\mathbf{E}}$$

∇^2: Laplacian operator

$$\nabla^2\tilde{\mathbf{E}} + \omega^2\mu\varepsilon_c\tilde{\mathbf{E}} = 0$$

$$\nabla^2\tilde{\mathbf{E}} - \gamma_c^2\tilde{\mathbf{E}} = 0$$
$$\nabla^2\tilde{\mathbf{H}} - \gamma_c^2\tilde{\mathbf{H}} = 0$$

$$\gamma_c^2 = -\omega^2\mu\varepsilon_c$$

Propagation constant

$$\nabla^2 \tilde{\mathbf{E}} - \gamma_c^2 \tilde{\mathbf{E}} = 0$$
$$\nabla^2 \tilde{\mathbf{H}} - \gamma_c^2 \tilde{\mathbf{H}} = 0$$

6개의 **Helmholtz equations**
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