Chapter 2. Continuous-time Signals and Systems I

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Continuous-time signals and systems I

The models of signals and systems

- Physical systems are modeled by mathematical equations. *e.g.*

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau = v(t)$$

or

$$y[n] - y[n-1] = Hx[n-1]$$

- Physical signals are modeled by mathematical functions. *e.g.*

$$\text{temperature at a point } = \theta(t)$$

or

$$x[n] = (0.3)^n u[n]$$

  - Continuous-time signals (continuous or discrete amplitude)
  - Discrete-time signals (continuous or discrete amplitude)

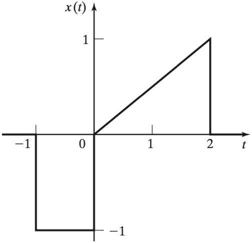
Continuous-time signals and systems I

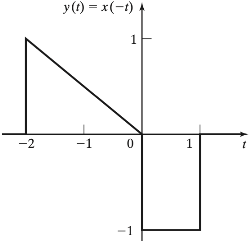
Time reversal

- The time reversal transformed signal $y(t)$ of $x(t)$ is obtained by replacing t with $-t$ in the original signal $x(t)$. That is,

$$y(t) = x(-t)$$

- The graph of the time-reversed signal $y(t)$ is the mirror image of the original signal $x(t)$, reflected about the vertical axis.





- Playing music on a CD backwards is an example of time reversal.

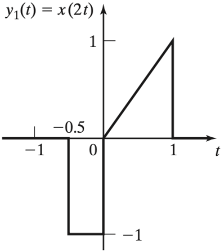
Time scaling

- A time-scaled version of $x(t)$ is

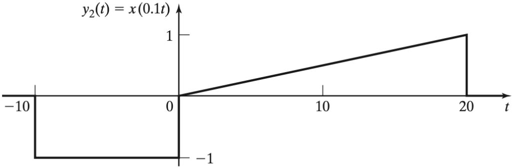
$$y(t) = x(at)$$

  where a is a real constant.
  - $|a| > 1 \rightsquigarrow$ compression in time
  - $|a| < 1 \rightsquigarrow$ expansion in time
- The graph of time-scaled signals

$a = 2$



$a = 0.1$



- A real-life example of time scaling
  - When you listen to an answering-machine message on fast forward, you can hear the voice pitch (frequency content of the speaker's voice) is increased. This is due to speeding up the signal in time.
  - When you play a forty-five-revolutions-per-minute (45-rpm) analog recording at 33rpm, you can hear the voice pitch is decreased. This is due to slowing down the signal in time.

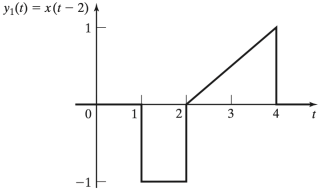
Time shifting

- A time-shifted version of $x(t)$ is

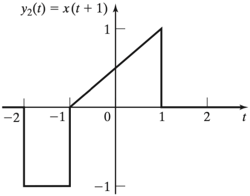
$$y(t) = x(t - t_0)$$

  where t_0 is a constant.
  - $t_0 > 0 \rightsquigarrow$ delayed in time
  - $t_0 < 0 \rightsquigarrow$ advanced in time
- The graph of time-shifted signals

$t_0 = 2$



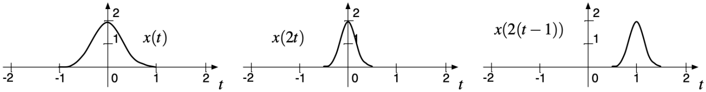
$t_0 = -1$



- May seem counterintuitive. Think about where the $t - t_0$ is zero.

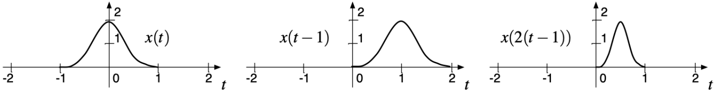
Combination of time transformation I

- Time scaling, shifting, and reversal can all be combined.[1]
- Operation can be performed in any order, but care is required.
- Example: $x(2(t-1))$ (time shifting and time scaling combined)
  - Scale first, then shift (Compress by 2, shift by 1) $\Rightarrow$ Correct

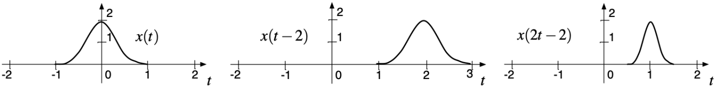


Combination of time transformation II

- Shift first, then scale (Shift by 1, compress by 2) $\Rightarrow$ Incorrect



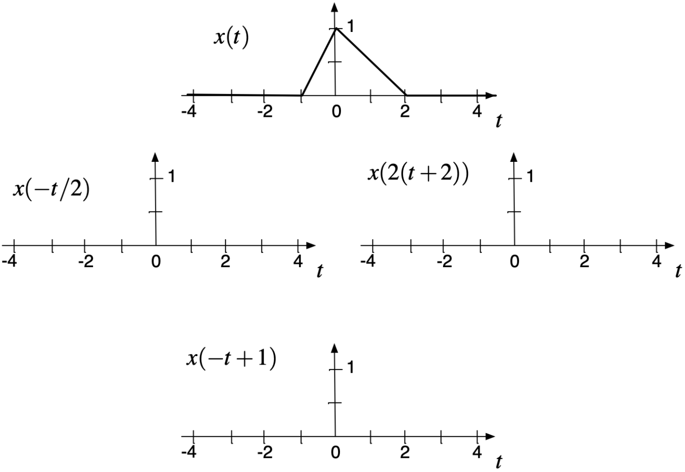
- Shift first, then scale (Rewrite $x(2(t-1)) = x(2t-2)$, Shift by 2, scale by 2) $\Rightarrow$ Correct



- Where is $2(t-1)$ equal to zero?

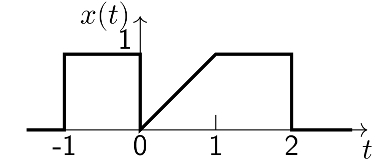
Combination of time transformation III

- Try these by yourselves



Combination of time transformation IV

- Another try



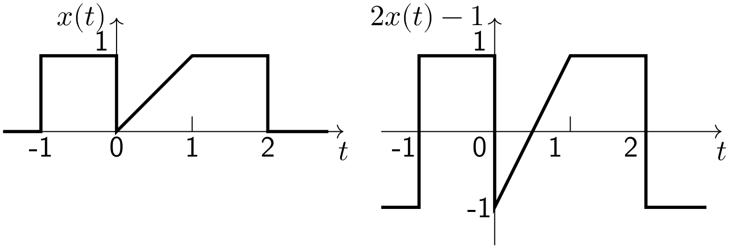
[1] The contents of this slide is borrowed from the lecture slide of Prof. Cuff's of Princeton University

Amplitude transformation

- The general form of amplitude transformation is

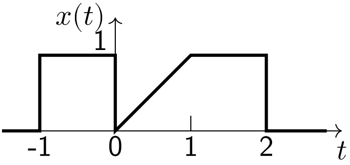
$$y(t) = Ax(t) + B$$

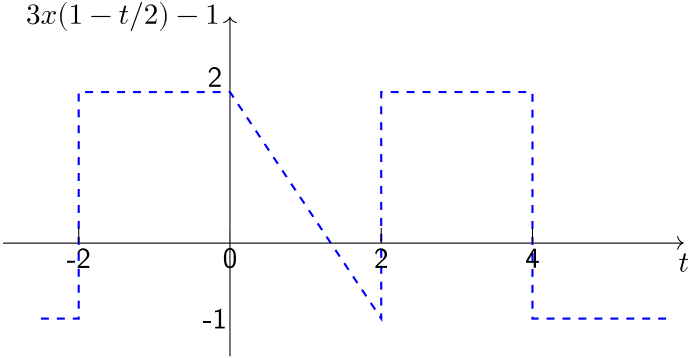
  - The negative sign of A inverts the signal
  - The absolute value of A determines the amplitude scaling
  - The value of B shifts the amplitude of the signal up or down



Time and amplitude transformation of a signal

- Try





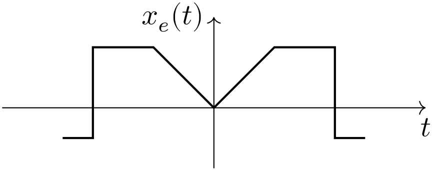
Summary: transformations of signals

Name	$y(t)$
Time reversal	$x(-t)$
Time scaling	$x(at)$
Time shifting	$x(t - t_0)$
Amplitude reversal	$-x(t)$
Amplitude scaling	$Ax(t)$
Amplitude shifting	$x(t) + B$

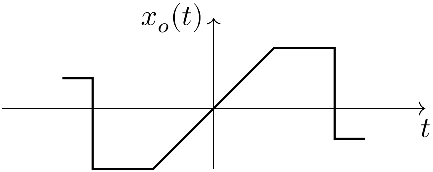
Continuous-time signals and systems I

Even and odd signals

- Even signal: $x_e(t) = x_e(-t)$



- Odd signal: $x_o(t) = -x_o(-t)$



- All signals can be represented as the sum of an even signal and an odd signal

$$x(t) = \frac{1}{2}\left[x(t) + x(-t)\right] + \frac{1}{2}\left[x(t) - x(-t)\right] = x_e(t) + x_o(t)$$

- The characterics of even and odd signals (Can you show them?)
  - The sum of two even signals is even.
  - The sum of two odd signals is odd.
  - The sum of an even signal and an odd signal is neither even nor odd.
  - The product of two even signals is even.
  - The product of two odd signals is even.
  - The product of an even signal and an odd signal is odd.

Periodic signals

Definition: periodic signal

A signal $x(t)$ is periodic if

$$\forall t, \quad x(t) = x(t+T), \quad T > 0$$

- The constant T is the period.
- By definition, a periodic function satisfies the equation

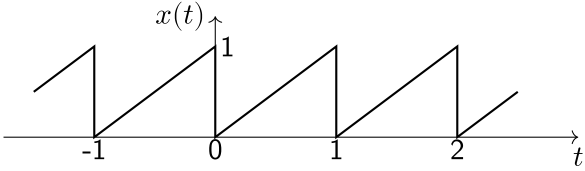
  $$x(t) = x(t+nT)$$

  where n is any integer.
- The minimum value of the period $T > 0$ that satisfies the above definition is called the fundamental period of the signal and denoted as T_0. The fundamental frequency is given by

  $$f_0 = \frac{1}{T_0}\text{Hz}, \quad \omega = 2\pi f_0 = \frac{2\pi}{T_0}\text{rad/s}$$

Examples of periodic signals

- $x_c(t) = \cos\omega t$ and $x_s(t) = \sin\omega t$
- The movement of a clock pendulum (it is *modeled* as sinusoids)
- A sawtooth wave



- $x(t) = e^{\sin t}$ is periodic? Yes. Because for $T = 2\pi$

$$x(t+T) = e^{\sin(t+2\pi)} = e^{\sin t} = x(t)$$

- $x(t) = te^{\sin t}$ is periodic? No. There is no T satisfying

$$x(t+T) = (t+T)e^{\sin(t+T)} \stackrel{?}{=} te^{\sin t} = x(t)$$

The sum of continuous-time periodic signals

$$x(t) = x_1(t) + x_2(t) + \cdots + x_N(t), \quad \text{where } x_i\text{'s are periodic}$$

- The sum of continuous-time periodic signals is periodic if and only if the ratios of the periods of the individual signals are ratios of integers.
- (If the sum is periodic), the fundamental period can be found as follows:
  1. Let T_{01} the period of the first signal and T_{0i} period of i-th signal $(2 \leq i \leq N)$
  2. Convert each period ratio, T_{01}/T_{0i} to an irreducible fraction
  3. Find k_0, the least common multiple of the denominator of the fractions
  4. Then, the fundamental period of the sum of signals is $T_0 = k_0 T_{01}$.

- Example

$$x(t) = x_1(t) + x_2(t) + x_3(t) = \cos(3.5t) + \sin(2t) + 2\cos\left(\frac{7t}{6}\right)$$

  - $x(t)$ is periodic?

$$T_{01} = \frac{2\pi}{\omega_1} = \frac{2\pi}{3.5},\ T_{02} = \frac{2\pi}{\omega_2} = \frac{2\pi}{2},\ T_{03} = \frac{2\pi}{\omega_3} = \frac{2\pi}{7/6}$$
$$\Rightarrow \frac{T_{01}}{T_{02}} = \frac{2}{3.5} = \frac{4}{7},\ \frac{T_{01}}{T_{03}} = \frac{7/6}{3.5} = \frac{1}{3}$$

    Both are ratios of integers; therefore, $x(t)$ is periodic.
  - What is fundamental period? Since the LCM of the denominators is 21,

$$T_0 = 21 \times \frac{2\pi}{3.5} = 12\pi(s)$$

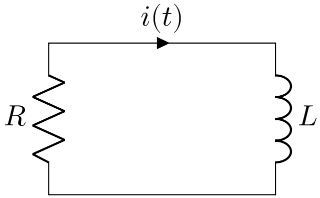
Continuous-time signals and systems I

Exponential function: introduction

- A signal whose time rate of change is directly proportional to the signal itself

$$\frac{dx(t)}{dt} = ax(t) \quad \Rightarrow \quad x(t) = Ce^{at}$$

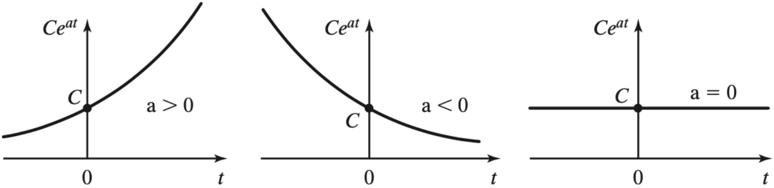
- RL circuit



$$L\frac{di(t)}{dt} + Ri(t) = 0 \quad \Rightarrow \quad \frac{di(t)}{dt} = -\frac{R}{L}i(t).$$

- Parameters C and a can be complex.

Ce^{at}: Both C and a real

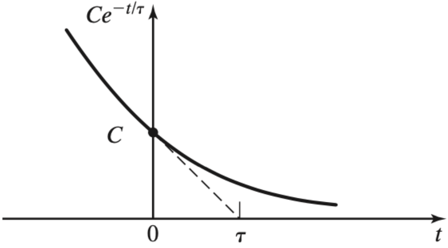


- For $a < 0$, we express the exponentials as

$$x(t) = Ce^{at} = Ce^{-t/\tau}, \quad \tau > 0$$

where τ is called the time constant of the exponential.

- The meaning of time constant



  - The derivative of $x(t)$ at $t = 0$ is given by

$$\left.\frac{dx(t)}{dt}\right|_{t=0} = \left.-\frac{C}{\tau}e^{-t/\tau}\right|_{t=0} = -\frac{C}{\tau}.$$

    If the signal continued to decay from $t = 0$ at this rate, it would be zero at $t = \tau$.
  - Actually, the value of the signal at $t = \tau$ is equal to $Ce^{-1} = 0.368C$. $\rightsquigarrow$ The signal has decayed to 36.8% of its amplitude after τ seconds.

Ce^{at}: C complex, a imaginary

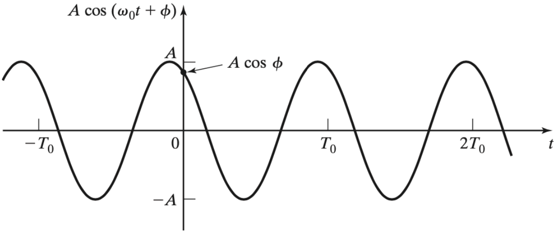
$$x(t) = Ce^{at};\ C = Ae^{j\phi} = A\angle\phi,\ a = j\omega_0$$

- $x(t)$ can be expressed as

$$\begin{aligned} x(t) &= Ae^{j\phi}e^{j\omega_0 t} = Ae^{j(\omega_0 t+\phi)} \\ &= A\cos(\omega_0 t + \phi) + jA\sin(\omega_0 t + \phi) \end{aligned}$$

  The sinusoids are periodic, so the $x(t)$ is also periodic

- A plot of the real part of $x(t)$



Harmonically related complex exponentials

Harmonically related complex exponentials are a set of functions with frequencies related by integers, of the form

$$x_k(t) = A_k e^{jk\omega_0 t}, \; k = \pm 1, \pm 2, \ldots$$

- The harmonically related complex exponentials will be used extensively when we study the Fourier series respresentation of periodic signals.

Ce^{at}: Both C and a complex

$$x(t) = Ce^{at};\ C = Ae^{j\phi},\ a = \sigma_0 + j\omega_0$$

- $x(t)$ can be expressed as

$$\begin{aligned} x(t) &= Ae^{j\phi}e^{(\sigma_0+j\omega_0)t} = Ae^{\sigma_0 t}e^{j(\omega_0 t+\phi)} \\ &= \underbrace{Ae^{\sigma_0 t}\cos(\omega_0 t+\phi)}_{Re[x(t)]} + j\underbrace{Ae^{\sigma_0 t}\sin(\omega_0 t+\phi)}_{Im[x(t)]} \end{aligned}$$

- A plot of the real part of $x(t)$