Chapter 10. Discrete-time Linear and Time-Invariant Systems

Kwangsoo Kim

Hanbat National University
Dept. Electronics Engineering

Discrete-time LTI Systems

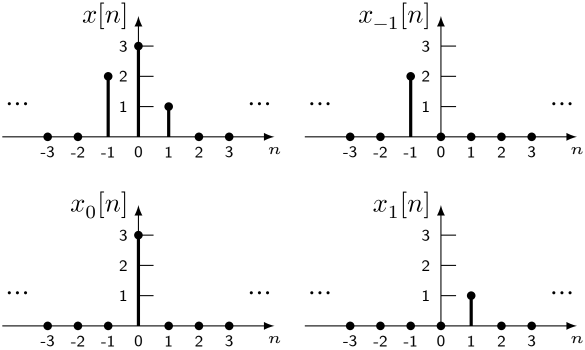
Introduction

- In one sense, discrete-time systems are easier to analyze and design
  - Difference equations are easier to solve than are differential equations
- In a different sense, discrete-time systems are more difficult to analyze and design
  - The system characteristics are periodic in frequency.
- In this course, we consider only discrete-time systems that are both linear and time invariant, which is referred to as discrete-time LTI systems.

Discrete-time LTI Systems

Impulse representation of discrete-time signals

- Example



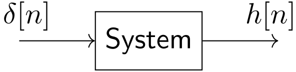
$$x[n] = x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1]$$

- In general,

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

Convolution sum

- Impulse reponse



$$\delta[n] \rightarrow h[n]$$

- The derivation for convolution sum
  - Time-invariance means $\delta[n-k] \rightarrow h[n-k]$
  - Linearity means $x[k]\delta[n-k] \rightarrow x[k]h[n-k]$
  - The signal can be expressed with impulse functions as $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$
  - Therefore,

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

$$\rightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]$$

Convolution sum with impulse function

- Convolution sum with impulse function

$$\delta[n] * g[n - n_0] = \delta[n - n_0] * g[n] = g[n - n_0]$$

- Do not confuse convolution with multiplication

$$\delta[n]g[n - n_0] = g[-n_0]\delta[n]$$

and

$$\delta[n - n_0]g[n] = g[n_0]\delta[n - n_0]$$

  - (Derivation) By definition,

$$\delta[n] * g[n - n_0] = \sum_{k=-\infty}^{\infty} g[k - n_0]\delta[n - k] = g[n - n_0]$$

Also,

$$\delta[n - n_0] * g[n] = \sum_{k=-\infty}^{\infty} \delta[k - n_0]g[n - k] = g[n - n_0]$$

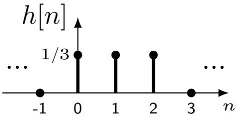
Impulse response (FIR)

- A moving-average filter which averages the last three inputs:

$$y[n] = (x[n] + x[n-1] + x[n-2])/3$$

- The impulse response for this system is

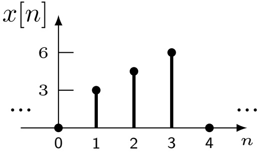
$$h[n] = (\delta[n] + \delta[n-1] + \delta[n-2])/3$$



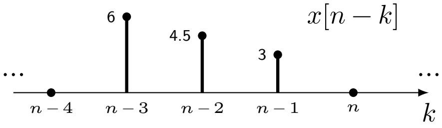
- This is a finite impulse response (FIR) system.

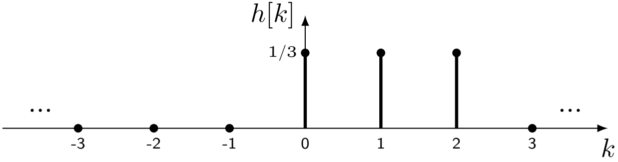
System response by convolution sum I

- For the system in the previous slide, let the input be



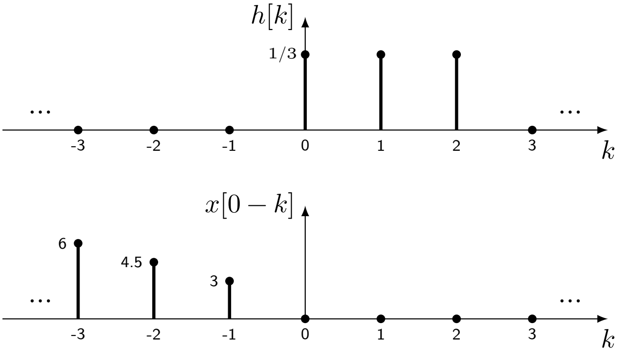
- The signal $x[n-k]$ and $h[k]$ are





System response by convolution sum II

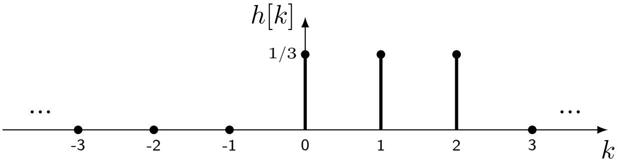
- For $n = 0$,

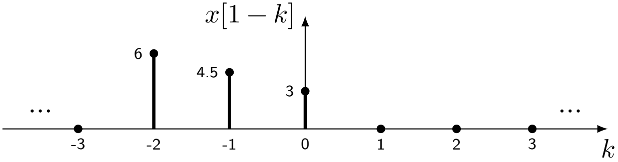


$$y[0] = \sum_{k=-\infty}^{\infty} x[0-k]h[k] = 0.$$

System response by convolution sum III

- For $n = 1$,

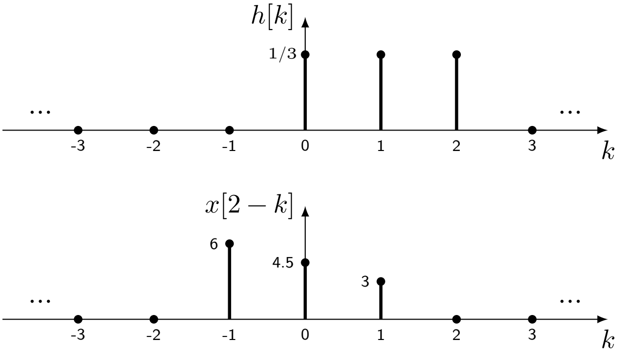




$$y[1] = \sum_{k=-\infty}^{\infty} x[1-k]h[k] = 1.$$

System response by convolution sum IV

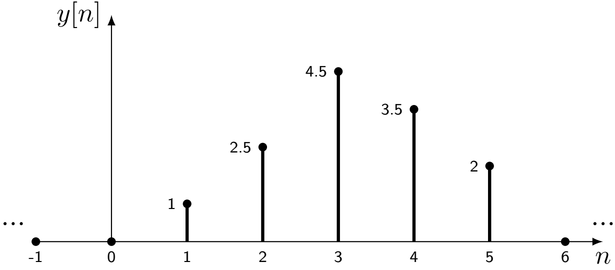
- For $n = 2$,



$$y[2] = \sum_{k=-\infty}^{\infty} x[2-k]h[k] = 4.5 \cdot 1/3 + 3 \cdot 1/3 = 2.5.$$

System response by convolution sum V

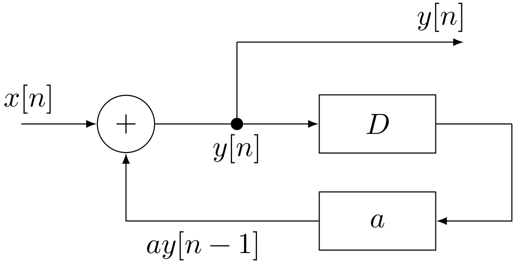
- Using the same procedure, we see that
  $y[3] = 4.5, y[4] = 3.5, y[5] = 2$, and $y[n] = 0$, for $n > 5$.



$$y[n] = \delta[n-1] + 2.5\delta[n-2] + 4.5\delta[n-3] \\ + 3.5\delta[n-4] + 2\delta[n-5].$$

Impulse response (IIR) I

- Consider the following system



$$y[n] = ay[n-1] + x[n]$$

Impulse response (IIR) II

- The impulse response can be obtained by applying a unit impulse function, with $x[0] = 1$ and $x[n] = 0, n \neq 0$.

$$\begin{aligned} y[0] &= h[0] = ay[-1] + x[0] = a(0) + 1 = 1, \\ y[1] &= h[1] = ay[0] + x[1] = a(1) + 0 = a, \\ y[2] &= h[2] = ay[1] + x[2] = a(a) + 0 = a^2, \\ y[3] &= h[3] = ay[2] + x[3] = a(a^2) + 0 = a^3, \\ &\vdots \end{aligned}$$

hence,

$$h[n] = \begin{cases} a^n, & n \geq 0 \\ 0, & n < 0 \end{cases} = a^n u[n]$$

- This is an infinite impulse response (IIR) system.

Step response of a discrete system

- Suppose that $h[n] = (0.6)^n u[n]$ and $x[n] = u[n]$. Then, the output signal is given by

$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} x[n-k]h[k] \\
&= \sum_{k=-\infty}^{\infty} u[n-k](0.6)^k u[k] \\
&= \sum_{k=0}^{n} (0.6)^k = \frac{1-(0.6)^{n+1}}{1-0.6} = 2.5[1-(0.6)^{n+1}], n \geq 0
\end{aligned}
$$

Properties of Convolution

- Commutative property

$$x[n] * h[n] = h[n] * x[n]$$

- Associative property

$$(f[n] * g[n]) * h[n] = f[n] * (g[n] * h[n])$$

- Distributive property

$$x[n] * h_1[n] + x[n] * h_2[n] = x[n] * (h_1[n] + h_2[n])$$

Discrete-time LTI Systems

Memory

- A memoryless (static) system is one whose current value of output depends on only the current value of input.
- Expanding the convolution sum for a memoryless system, it can be seen that

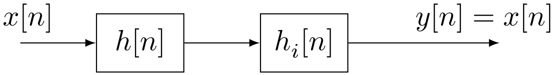
$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} x[n-k]h[k] \\
&= \cdots + x[n+2]h[-2] + x[n+1]h[-1] \\
&\quad + x[n]h[0] + x[n-1]h[1] + \cdots = h[0]x[n]
\end{aligned}
$$

That is, $h[n]$ must be zero for $n \neq 0 \rightsquigarrow h[n] = K\delta[n]$ where K is a constant.

Invertibility

- A system is invertible if its input can be determined from its output.
- A discrete-time LTI system is invertible if there exists a function $h_i[n]$ such that

$$h[n] * h_i[n] = \delta[n]$$



- The finding the impulse response $h_i[n]$ can be solved with the use of the z-transform

Causality

- A system is causal if the current value of the output depends on only the current value and past value of the input.
- Suppose that the input for a causal system is $\delta[n]$. Its output is $h[n]$, which must be zero for $n < 0$.
- The convolution sum for a causal LTI system can be expressed as

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[n-k]h[k] \\ &= x[n]h[0] + x[n-1]h[1] + x[n-2]h[2] + \cdots \\ &= \sum_{k=0}^{\infty} x[n-k]h[k] \end{aligned}$$

or, alternatively,

$$y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k].$$

Stability

- A system is BIBO stable if the output remains bounded for any bounded input.
- If $|x[n]| \leq M$,

$$
\begin{aligned}
|y[n]| &= \left| \sum_{k=-\infty}^{\infty} x[n-k]h[k] \right| \leq \sum_{k=-\infty}^{\infty} |x[n-k]h[k]| \\
&= \sum_{k=-\infty}^{\infty} |x[n-k]||h[k]| \leq \sum_{k=-\infty}^{\infty} M|h[k]| = M \sum_{k=-\infty}^{\infty} |h[k]|
\end{aligned}
$$

- It is sufficient that for $y[n]$ to be bounded,

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty \quad \text{(absolutely summable)}$$

  - This condition is also necessary condition.

Example for properties of LTI systems

- Let $h[n] = (\frac{1}{2})^n u[n]$.
- This system
  - has memory (is dynamic), since $h[n] \neq K\delta[n]$.
  - is causal, since $h[n] = 0$ for $n < 0$.
  - is stable, because

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 2 < \infty.$$

Unit step response

- The unit step response for an LTI system with the impulse response $h[n]$ is

$$s[n] = \sum_{k=-\infty}^{\infty} u[n-k]h[k] = \sum_{k=-\infty}^{n} h[k].$$

- For example, if the impulse response is $h[n] = (0.6)^n u[n]$, the unit step response is

$$s[n] = \sum_{k=0}^{n} (0.6)^k = \frac{1-(0.6)^{n+1}}{1-0.6} u[n] = 2.5(1-(0.6)^{n+1})u[n]$$

- Note that

$$s[n] - s[n-1] = \sum_{k=-\infty}^{n} h[k] - \sum_{k=-\infty}^{n-1} h[k] = h[n].$$

Discrete-time LTI Systems

Difference-equation models

- An LTI discrete-time systems are usually modeled by linear difference equations with constant coefficients.
- For example, a first-order linear difference equation is

$$y[n] = ay[n-1] + bx[n].$$

  where a and b are constants.
- The general form of an Nth-order linear difference equation with constant coefficients is, with $a_0 \neq 0$,

$$\begin{aligned} a_0 y[n] + a_1 y[n-1] + \cdots + a_N y[n-N] \\ = b_0 x[n] + b_1 x[n-1] + \cdots + b_M x[n-M] \end{aligned}$$

  or in a more compact form:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k], \quad a_0 \neq 0. \tag{1}$$

Solutions of difference equations

- The classical method for solutions of difference equation (1) require that the general solution be expressed as

$$y[n] = y_c[n] + y_p[n]$$

where $y_c[n]$ is called the complementary function (natural response) and $y_p[n]$ is a particular solution(forced response).

Natural response I

- The natural response satisfies the homogeneous equation

$$a_0 y[n] + a_1 y[n-1] + \cdots + a_N[n-N] = 0 \tag{2}$$

- Assume that the solution of this equation is of the form $y_c[n] = Cz^n$. Then,

$$\begin{aligned} y_c[n] &= Cz^n, \\ y_c[n-1] &= Cz^{n-1} = Cz^{-1}z^n, \\ y_c[n-2] &= Cz^{n-2} = Cz^{-2}z^n \\ y_c[n-N] &= Cz^{n-N} = Cz^{-N}z^n. \end{aligned}$$

- Substituting these terms into (2) yields

$$a_0 z^N + a_1 z^{N-1} + \cdots + a_{N-1} z + a_N = 0$$

which is called the characteristic equation.

Natural response II

- The characteristic equation may be factored as

$$a_0(z - z_1)(z - z_2)\cdots(z - z_N) = 0.$$

- If there is no repeated roots, the solution of eq. (2) may be expressed as

$$y_c[n] = C_1 z_1^n + C_2 z_2^n + \cdots + C_N z_N^n.$$

where $C_1, C_2, \ldots, C_N$ are evaluated later.

Forced response I

- The forced response is any function that satisfies eq. (1).
- Assume that the forced response is the sum of functions of the mathematical form of the excitation $x[n]$ and the delayed excitation $x[n-k]$ that differ in form from $x[n]$.
  - $x[n] = 3(0.2)^n \rightsquigarrow y_p[n] = P(0.2)^n$
  - $x[n] = n^3 \rightsquigarrow y_p[n] = P_1 + p_2 n + P_3 n^2 + P_4 n^3$
  - $x[n] = \cos 2n \rightsquigarrow y_p[n] = P_1 \cos 2n + P_2 \sin 2n$

Example for difference equation solution

$$y[n] - 0.6y[n-1] = x[n] = 4u[n]$$

- The characteristic equation is

$$z - 0.6 = 0 \quad \Rightarrow \quad z = 0.6$$

  So, the natural response is $y_c[n] = C(0.6)^n$.
- Because the forcing input is constant, the forced response is chosen as $y_p[n] = P$. Substituting this function into the difference equation yields

$$P - 0.6P = 4 \quad \Rightarrow \quad P = 10,$$

- So the general solution is

$$y[n] = y_c[n] + y_p[n] = C(0.6)^n + 10.$$

Comments on the solutions

- The natural response is dependent only on the structure of the system (the left side of eq. (1). It is also called the unforced response, or the zero-input response.
  - However, the value of unknown C_i's are the functions of both the excitation and the initial conditions.
- The forced response is a function of the system structure and of the excitation, but is independent of the initial conditions. It is also called the zero-state response (zero-state means zero initial condition).
- For almost all models of physical systems, the natural response goes to zero with increasing time, and then only the forced response remains (under the assumption of BIBO stability).
  $\rightsquigarrow$ The forced response is sometimes called the steady-state response and the natural response the transient response.

The natural response in case of repeated roots

- The natural response form should be changed in the general case of an rth-order root z_i in the characteristic equation. If the characteristic equation is factored into

$$a_0(z - z_i)^r(z - z_k)\cdots = 0$$

  The term in the natural response for this root is

$$(C_1 + C_2 n + C_3 n^2 + \cdots + C_r n^{r-1})z_i^n.$$

- For example, suppose that the characteristic equation of 4-th order difference equation is

$$(z - z_1)^3(z - z_2) = 0,$$

  then the natural response should be the form

$$y_c[n] = (C_1 + C_2 n + C_3 n^2)z_1^n + C_4 z_4^n.$$

Solution by iteration

- A difference equation can always be solved by iteration.

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$

  with the initial conditions $y[m], y[m+1], \ldots, y[m+N-1]$.

- For example, let $m = 0$ and you get

$$\begin{aligned} y[N] = &-a_1 y[N-1] - a_2 y[N-2] - \cdots - a_N y[0] \\ &+ b_0 x[N] + b_1 x[N-1] + \cdots + b_M x[N-M] \end{aligned}$$

  and then $y[N+1], y[N+2], \ldots$.

- The iterative solution does not result in $y[n]$ as an explicit function of n (like the classical solution), but is easily implemented on a digital computer (system simulation).

Terms in the natural response

- The general term in the natural response is given by $C_i z_i^n$, where z_i^n is called a system mode.
- When z_i is real and positive, let $z_i = e^{\Sigma_i}$ with Σ_i real, and

$$C_i z_i^n = C_i (e^{\Sigma_i})^n = C_i e^{\Sigma_i n}$$

- When z_i is complex, let $z_i = e^{\Sigma_i + j\Omega_i}$. Since the natural response must be real, two of the terms of $y_c[n]$ can be expressed as

$$\begin{aligned} C_i z_i^n + C_i^* (z_i^*)^n &= |C_i| e^{j\beta_i} e^{\Sigma_i n} e^{j\Omega_i n} + |C_i| e^{-j\beta_i} e^{\Sigma_i n} e^{-j\Omega_i n} \\ &= 2|C_i| e^{\Sigma_i n} \cos(\Omega_i n + \beta_i). \end{aligned}$$

Stability

- The general solution is given by

$$y[n] = y_c[n] + y_p[n]$$

- $y_p[n]$ is of the mathematical form as the system input $x[n]$. $\rightsquigarrow$ If $x[n]$ is bounded, $y_p[n]$ is also bounded.
- The general term of $y_c[n]$ is $C_i z_i^n$, where z_i is a root of the sytem characteristic equation. $\rightsquigarrow$ The magnitude of this term is given by $|C_i||z_i|^n$. $\rightsquigarrow$ If $|z_i| < 1$, the magnitude of the term approaches zero as $n \to \infty$.
- Thus, the necessary and sufficient condition that a causal discrete-time LTI system is BIBO stable is that $|z_i| < 1$.

Discrete-time LTI Systems

Linearity

- Consider two real inputs and outputs for an LTI system, that is,

$$x_i[n] \to y_i[n], \quad i = 1, 2$$

- Because of the linearity,

$$a_1 x_1[n] + a_2 x_2[n] \to a_1 y_1[n] + a_2 y_2[n]$$

- If we choose $a_1 = 1$ and $a_2 = j = \sqrt{-1}$, then

$$x_1[n] + j x_2[n] \to y_1[n] + j y_2[n].$$

- For a complex input function to an LTI system, the real part of the input produces the real part of the output, and the imaginary part of the input produces the imaginary part of the output.

Complex inputs for LTI systems I

- Consider the steady-state system response to the complex-exponential input

$$x[n] = Xz^n \tag{3}$$

where X and z can be complex.

- The LTI system is modeled by Nth-order difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{N} b_k x[n-k] \tag{4}$$

- The forced response (steady-state response) is of the same mathematical form. Hence

$$y_{ss}[n] = Yz^n \tag{5}$$

Complex inputs for LTI systems II

- Substituting eq. (3) and (5) into eq. (4) results in

$$(a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}) Y z^n$$
$$= (b_0 + b_1 z^{-1} + \cdots + b_N z^{-N}) X z^n$$

, from which Y is given by

$$Y = \left[\frac{b_0 + b_1 z^{-1} + \cdots + b_N z^{-N}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}} \right] X = H(z) X.$$

Here, $H(z)$ is called a transfer function.

- In summary,

$$x[n] = X z_1^n \to y_{ss}[n] = X H(z_1) z_1^n.$$

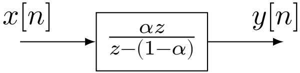
Example for transfer function

- The difference equation of the α-filter

$$y[n] - (1 - \alpha)y[n-1] = \alpha x[n]$$

- The transfer function

$$H(z) = \frac{\alpha}{1 - (1 - \alpha)z^{-1}} = \frac{\alpha z}{z - (1 - \alpha)}$$



Sinusoidal inputs for LTI systems

- Consider the steady-state system response to the sinusoidal input

$$x[n] = Xz_1^n = |X|e^{j\phi}e^{j\Omega_1 n}$$
$$= |X|\cos(\Omega_1 n + \phi) + j|X|\sin(\Omega_1 n + \phi)$$

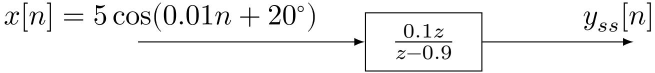
- Let $H(z_1) = |H(e^{\Omega_1})|e^{j\theta_H}$, the steady-state output is

$$y_{ss}[n] = XH(e^{j\Omega_1})e^{j\Omega_1 n} = |X||H(e^{j\Omega_1})|e^{j(\Omega_1 n + \phi + \theta_H)}$$
$$= |X||H(e^{j\Omega_1})|\left[\cos(\Omega_1 n + \phi + \theta_H) + j\sin(\Omega_1 n + \phi + \theta_H)\right]$$

- Since the real part of the input signal produces the real part of the output signal,

$$|X|\cos(\Omega_1 n + \phi) \rightarrow |X||H(e^{j\Omega_1})|\cos(\Omega_1 n + \phi + \theta_H).$$

Example for sinusoidal response



- Calculate the value of transfer function of $z = e^{j\Omega} = e^{j0.01}$

$$
\begin{aligned}
H(z)\Big|_{z=e^{j0.01}} &= \frac{0.1(e^{j0.01})}{e^{j0.01} - 0.9} \\
&= \frac{0.1\angle 0.573^\circ}{0.99995 + j0.01 - 0.9} \\
&= \frac{0.1\angle 0.573^\circ}{0.1004\angle 5.71^\circ} = 0.996\angle -5.14^\circ
\end{aligned}
$$

- So the steady-state output is

$$
\begin{aligned}
y_{ss}[n] &= 5(0.996)\cos(0.01n + 20^\circ - 5.14^\circ) \\
&= 4.98\cos(0.01n + 14.86^\circ)
\end{aligned}
$$

Impulse response and transfer function

- For $x[n] = z^n$, the system response is

$$y_{ss}[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} h[k]z^{n-k} = z^n \sum_{k=-\infty}^{\infty} h[k]z^{-k}$$

- From the system response for complex exponential input,

$$y_{ss}[n] = H(z)z^n = z^n \sum_{k=-\infty}^{\infty} h[k]z^{-k}$$

So,

$$H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$$

Homeworks

- Problems at the end of the chapter
  - 10.6
  - 10.11
  - 10.14
  - 10.19
  - 10.23 (a)
  - 10.25
  - 10.34 (a) (b)